4AESE - Analyse des Systèmes Non-Linéaires

Chapitre 5 : State Feedback Stabilization

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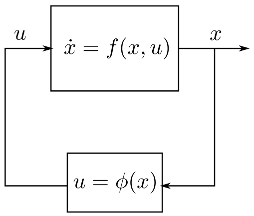

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE
LAAS CNRS

version : 9-2023

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Problem statement

Consider a system

$$\dot{x} = f(x, u)$$

State feedback stabilization problem

Design a control law $u = \phi(x)$ such that the origin $x = 0$ is an asymptotically stable equilibrium point for the closed-loop system

$$\dot{x} = f(x, \phi(x))$$

- $u = \phi(x)$ is a *static feedback*, a memoryless function of x
- *Dynamic feedback* : $u = \phi(x, z)$, with z a state of a dynamic system $\dot{z} = g(x, z)$

A different equilibrium point may be stabilized : x_{eq}

$\rightarrow$ It requires the existence of a steady-state control u_{eq} such that

$$0 = f(x_{eq}, u_{eq})$$

Apply the change of variable

$$x_\delta = x - x_{eq} \qquad \text{and} \qquad u_\delta = u - u_{eq}$$

and we have

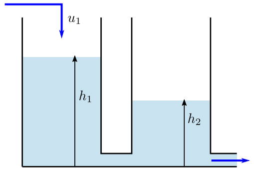
$$\dot{x}_\delta = f(x_{eq} + x_\delta, u_{eq} + u_\delta) \triangleq f_\delta(x_\delta, u_\delta)$$

with $f_\delta(0,0) = 0$.

- the previous formulation is retrieved
- the control $u_\delta = \phi(x_\delta)$ is to be designed
- then, the overall control is $u = \phi(x_\delta) + u_{eq}$

Example

Consider a hydraulic system with two tanks.



Dynamical model of liquid levels :

$$\begin{cases} \dot{h}_1 = 0.01 u_1 - 0.05\,\text{sign}(h_1 - h_2)\sqrt{20|h_1 - h_2|} \\ \dot{h}_2 = 0.05\,\text{sign}(h_1 - h_2)\sqrt{20|h_1 - h_2|} - 0.05\sqrt{20\,h_2} \end{cases}$$

Desired liquid level $h_{1eq} = 0.9m$

- At the equilibrium :

$$\begin{cases} u_1 = & 5\sqrt{20|h_1 - h_2|} \\ h_1 = & 2h_2 \end{cases} \quad \Rightarrow \quad \begin{cases} u_{1eq} = & 15 \\ h_{2eq} = & 0.45 \end{cases}$$

- Defining $x = h - h_{eq}$ and $u = u_1 - u_{1eq}$, new system :

$$\begin{cases} \dot{x}_1 = 0.01u + 0.15 - 0.05\,\text{sign}(x_1 - x_2 + 0.45)\sqrt{20|x_1 - x_2 + 0.45|} \\ \dot{x}_2 = 0.05\,\text{sign}(x_1 - x_2 + 0.45)\sqrt{20|x_1 - x_2 + 0.45|} - 0.05\sqrt{20\,(x_2 + 0.45)} \end{cases}$$

- with the equilibrium point at the origin $x = 0$ and $u = 0$.

Linearization

For linear time invariant systems

$$\dot{x} = Ax + Bu$$

State feedback control : $u = -Kx$

- Resulting closed-loop system

$$\dot{x} = \Big(A - BK\Big)x$$

- Closed-loop system asymptotically stable iff $A - BK$ is Hurwitz
- Several systematic methods to design gain K

Linearization

As for input free systems, nonlinear systems can be linearized around $(x = 0\ ,\ u = 0)$ (equilibrium point)

$$\dot{x} = f(x,u) \qquad \approx \qquad \dot{x} = Ax + Bu$$

with

$$A = \frac{\partial f}{\partial x}(x,u)\Big|_{x=0,u=0} \qquad \text{and} \qquad B = \frac{\partial f}{\partial u}(x,u)\Big|_{x=0,u=0}$$

- A linear state feedback $u = -Kx$ can be designed with linear tools.
- The origin is still an equilibrium point for the closed-loop system

$$\dot{x} = f(x, -Kx)$$

- For a small enough x, the origin is *locally stabilized*.
- A Lyapunov function may be used to estimate the region of attraction

Example

We want to stabilize the scalar system

$$\dot{x} = x^2 + u \quad \xrightarrow{\text{linearization}} \quad \dot{x} = u$$

easily stabilized by control law $u = -kx$, $k > 0$

- The origin is still an equilibrium point for the closed-loop system
$$\dot{x} = -kx + x^2$$
- For a small enough x, the origin is asymptotically stable.
- With the Lyapunov function $V = \frac{1}{2}x^2$, an estimation of the region of attraction is the set $\{|x| < k\}$
- Actually, the region of attraction is the set $\{x < k\}$

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Introductory example

Back to the previous scalar system

$$\dot{x} = x^2 + u$$

- The linear state feedback $u = -kx$, $k > 0$, ensures local asymptotic stability
$$\dot{x} = -kx + x^2 \qquad \text{(closed-loop)}$$
- The nonlinear state feedback
$$u = -kx - x^2, \qquad k > 0,$$
ensures global stabilization
$$\dot{x} = -kx \qquad \text{(closed-loop} \to \text{linearized)}$$

↪ The control has canceled the nonlinearity

Feedback linearization

Consider nonlinear system of the form (affine in u)

$$\dot{x} = f(x) + g(x)u \qquad \text{with } f(0) = 0,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

Assume a diffeomorphism $T(x)$ on a set[1] D, with $T(0) = 0$, such that the change of variable transforms the system into

$$\dot{z} = Az + B\Big[\psi(x) + \gamma(x)u\Big] \qquad \text{with } \gamma(x) \text{ a nonsingular matrix } \forall x \in D$$

The nonlinear state feedback

$$u = \gamma^{-1}(x)\Big(-\psi(x) + v\Big)$$

cancels the nonlinearity and converts the system into

$$\dot{z} = Az + Bv$$

↪ a linear system with a new control variable v

1. Let D be a domain of $\mathbb{R}^n$ including the origin

- The origin $z = 0$ can be stabilized by (exponentially stable)

$$v = -Kz$$

- In $x-$coordinates, the control becomes

$$u = \gamma^{-1}(x)\Big(-\psi(x) - KT(x) \Big)$$

- It can be shown that the $x-$coordinates dynamic also has the exponential stability property in the neighborhood of $x = 0$
- Feedback linearization is based on exact mathematical cancellation of nonlinear terms

  ↪ requires a very good knowledge of the model

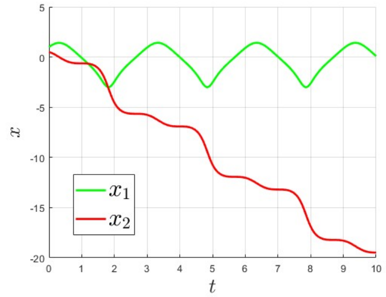
- Some nonlinear terms may be "good" terms and are helpful for stabilization

Example 1

Consider system

$$\dot{x} = \begin{bmatrix} a\sin(x_2) \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{with } a > 0$$

- On $D = \{|x_2| < \pi/2\}$, the origin is the unique equilibrium point $f(0,0) = 0$
- Open loop simulation with $u = 0$ (with $a = 5$ and $x_0 = [1\ 0.5]^T$)



Design the control law

- Change of variables

$$z = T(x) = \begin{bmatrix} x_1 \\ a\sin(x_2) \end{bmatrix} \qquad T(x) \text{ being a diffeomorphism on } D$$

- New system

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} a\cos(x_2)\Big(-x_1^2 + u \Big)$$

- with control law $u = x_1^2 + \frac{1}{a\cos x_2} v$, we have

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

  ↪ easy to place poles (it's a control companion form)

Simulations

In $z-$coordinates, control law :

$$v = -\underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{K} z$$

Results in the closed-loop system

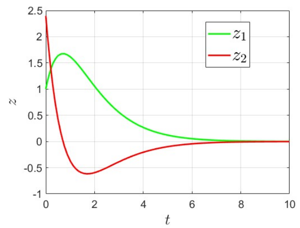
$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} z$$

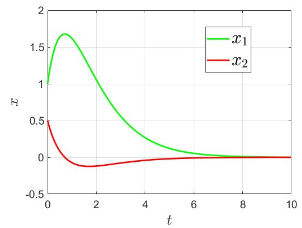
In $x-$coordinates, control law :

$$u = x_1^2 - \frac{1}{a\cos x_2} KT(x)$$

Results in the closed-loop system

$$\dot{x} = \begin{bmatrix} a\sin(x_2) \\ -\frac{1}{a\cos x_2} KT(x) \end{bmatrix}$$





($a = 5$ and $x_0 = [1\ 0.5]^T$)

Example 2

Consider system

$$\dot{x} = ax - bx^3 + u \qquad \text{with } a,\ b > 0$$

- First stabilizing state feedback $u = -(k+a)x + x^3$

  $\hookrightarrow$ closed-loop $\to \dot{x} = -kx$

- Second stabilizing state feedback $u = -(k+a)x$

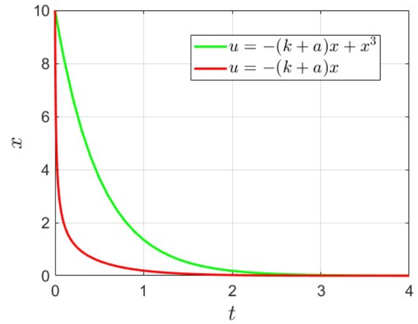
  $\hookrightarrow$ closed-loop $\to \dot{x} = -kx - bx^3$

- Lyapunov analysis : $V = \frac{1}{2}x^2$

$$\dot{V} = x\Big(-kx - bx^3\Big) = -kx^2 - bx^4 < 0$$

$\Rightarrow$ global asymptotic stability

Simulations



($a = b = 1$, $k = 2$ and $x_0 = 10$)

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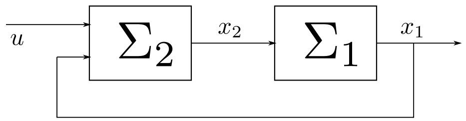
Backstepping

It is a nonlinear state feedback control design method

Consider nonlinear system of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)u \end{cases} \qquad \text{with } x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R},\ u \in \mathbb{R}$$

Sort of cascade connection of two subsystems



objective : design a state feedback to stabilize the origin

Design method

First, consider x_2 as a virtual control input for the first equation

- Let us assume a stabilizing control $x_2 = \phi(x_1)$ is known, with $\phi(0) = 0$ that is the origin of
$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\phi(x_1)$$
is asymptotically stable

- Assume also a Lyapunov function $V_1(x_1)$ is known, with
$$\frac{\partial V_1}{\partial x_1}\left[f_1(x_1) + g_1(x_1)\phi(x_1)\right] \leq -W(x_1) \qquad W(x_1) \text{ is positive definite}$$

- Let rewrite the original first equation as
$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\phi(x_1) + g_1(x_1)\left[x_2 - \phi(x_1)\right]$$

- And define the change of variable $z = x_2 - \phi(x_1)$

New formulation of the whole system
$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)\phi(x_1) + g_1(x_1)z \\ \dot{z} = F(x_1, x_2) + g_2(x_1, x_2)u \end{cases}$$
with $F(x_1, x_2) = f_2(x_1, x_2) - \dfrac{\partial \phi}{\partial x_1}\left[f_1(x_1) + g_1(x_1)\phi(x_1) + g_1(x_1)z\right]$

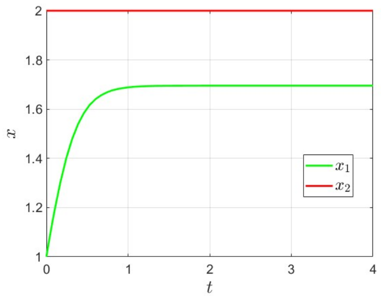
- Appears complicated... but the 1st equation is AS when $z = 0$
- Consider Lyapunov function candidate $V(x_1, x_2) = V_1(x_1) + \frac{1}{2}\big(\overbrace{x_2 - \phi(x_1)}^{z}\big)^2$
$$\dot{V} \leq -W(x_1) + z\left[\frac{\partial V_1}{\partial x_1}g_1(x_1) + F(x_1, x_2) + g_2(x_1, x_2)u\right]$$
- If $g_2 \neq 0$, choosing $u = -\dfrac{1}{g_2(x_1, x_2)}\left[\dfrac{\partial V_1}{\partial x_1}g_1(x_1) + F(x_1, x_2) + kz\right]$ yields
$$\dot{V} \leq -W(x_1) - kz^2 \qquad \text{for some } k > 0$$

$\Rightarrow$ it proves that the origin is asymptotically stable

Example

Consider system
$$\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = u \end{cases}$$

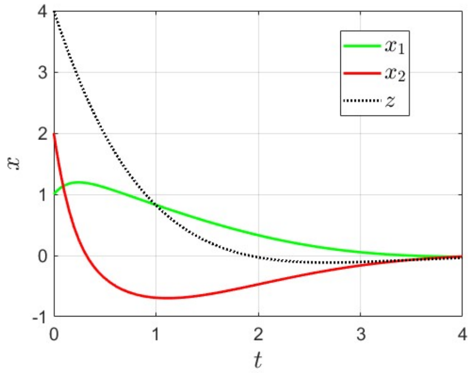
- Infinite number of equilibrium points
- Open loop simulation with $u = 0$ and $x_0 = [1\ 2]^T$



Objective : stabilize the origin with backstepping control

- Let's start with the first equation, $x_1 = 0$ stabilized with virtual control law
$$x_2 = \phi(x_1) \triangleq -x_1^2 - x_1$$
- Proved with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2 \quad \Rightarrow \quad \dot{V}_1 = -x_1^2 - x_1^4$
- Change of variable $z = x_2 + x_1^2 + x_1$ $(= x_2 - \phi(x_1))$
- That transforms the system into
$$\begin{cases} \dot{x}_1 = -x_1 - x_1^3 + z \\ \dot{z} = u + (1 + 2x_1)(-x_1 - x_1^3 + z) \end{cases}$$
- Consider Lyapunov function candidate $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$ for the overall system
$$\dot{V} = -x_1^2 - x_1^4 + z\Big(x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z) + u\Big)$$
- The origin $x = 0$ stabilized with control
$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z) - z$$

Simulations



Initial condition : $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

More general form

By recursive application of backstepping, one can consider *strict-feedback systems* of the form

$$\begin{cases} \dot{\eta} &= f_0(\eta) + g_0(\eta)x_1 \\ \dot{x}_1 &= f_1(\eta, x_1) + g_1(\eta, x_1)x_2 \\ \dot{x}_2 &= f_2(\eta, x_1, x_2) + g_2(\eta, x_1, x_2)x_3 \\ &\vdots \\ \dot{x}_{k-1} &= f_{k-1}(\eta, x_1, \dots, x_{k-1}) + g_{k-1}(\eta, x_1, \dots, x_{k-1})x_k \\ \dot{x}_k &= f_k(\eta, x_1, \dots, x_k) + g_k(\eta, x_1, \dots, x_k)u \end{cases}$$

where $\eta \in \mathbb{R}^n$, x_i are scalars, f_i equal 0 at the origin and $g_i \neq 0$ in some domain