Part I: Lyapunov Stability Theory (5 sessions)

Part II: Describing Function Method (3 sessions)

Associated tutorials: Vincent MAHOUT

Main references:

- Lecture notes of different colleagues C. Prieur & S. Tarbouriech, F. Gouaisbaut.
Lyapunov Stability Theory - Describing Function Method

Germain Garcia

Lyapunov Stability Theory
Summary

1. Introduction to stability
2. Stability in the sense of Lyapunov
3. Lyapunov method
4. Linear systems and linearization
5. Time-Varying Systems: the linear case
6. Input to State Stability (ISS)
Table of Contents

1. Introduction to stability
2. Stability in the sense of Lyapunov
3. Lyapunov method
4. Linear systems and linearization
5. Time-Varying Systems: the linear case
6. Input to State Stability (ISS)
Introduction to stability
Stability

- The notion of stability plays a key role in control theory and automation.
- There exists several notions of stability in control theory.
  - stability of an equilibrium point,
  - input-output stability.
- In this course, we are concerned with the stability of an equilibrium point, or internal stability.
- The internal stability will be studied thanks to the Lyapunov theory as we will see later.
Table of Contents

1. Introduction to stability

2. Stability in the sense of Lyapunov

3. Lyapunov method

4. Linear systems and linearization

5. Time-Varying Systems: the linear case

6. Input to State Stability (ISS)
Stability in the sense of Lyapunov
Stability of autonomous systems (1)

- Consider nonlinear systems without input, i.e. systems of the form:

\[
\dot{x}(t) = f(x(t)),
\]

where \( f \) is locally Lipschitz in a domain \( D \).

**Definition (Equilibrium points)**

Let us consider the system without input (1). One denotes an equilibrium point \( x_e \), as the solution of the equation

\[
f(x_e) = 0,
\]
Stability of autonomous systems (2)

- If the autonomous system (1) admits an equilibrium point, then

\[ \dot{x}(t) = f(t, x(t)), \quad x(0) = x_e \]

admits a unique solution \( x(t) = x_e, \forall t \geq 0 \). In other word, if the system is initialized at an equilibrium point, it remains on it.

- In this part, the notion of stability refers to the stability of an equilibrium point.
- By the change of variable

\[ \tilde{x}(t) = x(t) - x_e \]

one can always boil down to the study of the equilibrium point 0.
- In the sequel, one assumes that the equilibrium point studied is 0.
Definitions of stability (1)

- We are interested in the behavior of the solution $x(t)$ when $x(0) \neq 0$ but close to 0.

**Definition (Stability)**

The equilibrium point 0 is said to be **stable** if and only if

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0.$$ 

Different notions of stability

- Asymptotic stability
- Exponential stability
- Local and global stability
Definition of stability (1)

The solutions remain bounded as soon as the initial condition is small, as shown in the following figure:

\[ x(t) \text{ remains bounded as long as } x(0) \text{ is small.} \]
Definition of asymptotic stability (2)

- We are interested in the convergence of the solution $x(t)$ toward the equilibrium point 0, when the initial condition is not too far from 0.

**Definition (Attractivity)**

The equilibrium point 0 is said to be an attractor if and only if

$$\exists \delta > 0, \|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0,$$

$$\exists \delta > 0, \|x(0)\| < \delta \Rightarrow \forall \epsilon > 0, \exists t_1 > 0, \forall t > t_1, \|x(t)\| < \epsilon$$
Definition of asymptotic stability (2)

- Attractivity signifies that each solution initialized close enough to 0 converges toward 0 as $t \to \infty$
Definition of asymptotic stability (2)

- By using the notions of stability and attractivity, we can define the notion of asymptotic stability.

**Definition (Asymptotic stability)**

The equilibrium point 0 is said to be asymptotically stable if and only if it is stable and attractor.
The notions of stability and attractivity are two different notions.

- The stability notion relies to the system behavior when the initial condition is perturbed.
- The attractivity notion considers the convergence of solutions toward 0 when the initial conditions are close to 0.
We consider two examples to detail the difference between the notion of attractor and of stability.

- Example of a system globally attractor but unstable
- Example of a stable system but not attractor
Example of a system globally attractor but unstable

- Consider the following modified version of the butterfly system:

\[
\begin{align*}
\dot{x}_1(t) &= x_1^2(t)(x_2(t) - x_1(t)) + x_5^2(t) \\
\dot{x}_2(t) &= x_2^2(t)(x_2(t) - 2x_1(t))
\end{align*}
\]

- The unique equilibrium point is \( x_e = [0 \ 0]^T \).
- All the system trajectories converge toward \( x_e \). \( x_e \) is a global attractor.
Example of a system globally attractor but unstable

- The equilibrium point 0 is unstable as shown in the figure which follows.
- Actually, let us consider an initial condition close to 0, the trajectory $x(t)$ may be very large.
Example of a stable system but not attractor

- Consider the following example

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\sin(x_1(t))
\end{align*}
\]

- Compute the equilibrium points.
- Show that an equilibrium point \( x_e = [2k\pi, 0]^T \) is stable but not attractor.
Example of a stable system but not attractor

- The origin is stable but not an attractor.
Exponential stability

Definition (Exponential stability)

The equilibrium point \( x_e = 0 \) is **exponentially stable** if there exist two strictly positive scalars \( \alpha \) and \( k \) such that

\[
\exists \delta > 0, \|x(0)\| < \delta, \forall t \geq 0, \|x(t)\| < k\|x(0)\|e^{-\alpha t}
\]

Consider the system described

\[
\dot{x}(t) = -(1 + \sin^2(x(t)))x
\]

We have

\[
x(t) = x(0)e^{-\int_0^t (1+\sin^2(\tau))d\tau}
\]

and

\[
\|x(t)\| < \|x(0)\|e^{-t} \quad \text{because} \quad \int_0^t (1 + \sin^2(\tau))d\tau > t \quad (\alpha = k = 1)
\]
Consider now the system

\[ \dot{x}(t) = -x^2(t), \quad x(0) = 1 \]

We have

\[ x(t) = \frac{1}{1 + t} \]

\( \forall \alpha > 0, \text{ there exists } t_0 \text{ such that } t > t_0, \|x(t)\| > e^{-\alpha t} \)

We can see that from the definition of exponential stability, there exists \( \tau \) such that \( k = e^{\alpha \tau} \) and then we have

\[ \frac{\|x(t)\|}{\|x(0)\|} < e^{\alpha (\tau - t)} \]

and for \( t \geq \tau + \frac{3}{\alpha}, \frac{\|x(t)\|}{\|x(0)\|} \leq 5\% \).

We recover the notion of time constant.
Local stability - Global stability

All the definitions of stability are defined in terms of neighborhoods. Consequently the associated concepts are local and we speak of local stability, meaning that they are satisfied for initial conditions belonging to a proper subset of $\mathbb{R}^n$. This subset is called the domain of attraction of the considered equilibrium point. Its exact determination is in general very complex.

When the domain of attraction is $\mathbb{R}^n$, the equilibrium point is said globally stable, globally asymptotically stable or globally exponentially stable, depending of the corresponding stability concept associated to the equilibrium point.

For linear time-invariant systems, when a system is asymptotically stable, it is globally exponentially stable.

$$\dot{x}(t) = Ax, \quad x(0)$$

$$x(t) = x(0)\exp(At) = x(0)\sum_{i=1}^{n} w_i v_i \exp(\lambda_i)$$

and

$$\|x(t)\| \leq \left( \sum_{i=1}^{n} \|w_i\| \|v_i\| \right) \|x(0)\| \exp(\max_i \lambda_i) = k \|x(0)\| \exp(\max_i \lambda_i)$$
Lyapunov Method
Introduction (1)

- The main tool we use is the Lyapunov theory.

- **Aleksandr Mikhailovich Lyapunov (1857–1918)**, defended his PhD thesis *On Stability of Elliptic Equilibrium Forms of a Rotating Liquid*, in 1885, At the University of St Petersburg, with the advisor Pafnuti Lvovich Tchebychev (1821–1894). Tchebychev who also supervised the PhD of Andrei Andreyevich Markov (1856–1922).
Introduction (2)

- The second method of Lyapunov is a very interesting tool to deal with nonlinear systems.

  ▶ The objective is to study the convergence of the system trajectories towards the origin (equilibrium point of interest) without explicit description of these trajectories.

  ▶ To do this, we use functions with definite sign in a convex region $\mathcal{D}$ of the state space including the origin.

  ▶ Lyapunov stability theory enables determining stability without solving any differential equations.
Functions with definite sign

Let us define functions with definite sign in a convex region $\mathcal{D}$ of the state space including the origin.

- $V(x)$ is positive definite in $\mathcal{D}$ if $V(x) > 0$ for any $x \in \mathcal{D}$, $x \neq 0$ and $V(0) = 0$.
- $V(x)$ is positive semi-definite in $\mathcal{D}$ if $V(x) \geq 0$ for any $x \in \mathcal{D}$, $x \neq 0$ and $V(0) = 0$.
- $V(x)$ is negative definite in $\mathcal{D}$ if $V(x) < 0$ for any $x \in \mathcal{D}$, $x \neq 0$ and $V(0) = 0$.
- $V(x)$ is negative semi-definite in $\mathcal{D}$ if $V(x) \leq 0$ for any $x \in \mathcal{D}$, $x \neq 0$ and $V(0) = 0$.

Remark: $V(x)$ negative semi-definite $\Leftrightarrow$ $-V(x)$ positive semi-definite

Examples:
- $V(x) = (x_1 + x_2)^2$ is positive semi-definite in $\mathbb{R}^2$.
- $V(x) = x_1^2 + x_2^2$ is positive definite in $\mathbb{R}^2$. ($x^TPx$, $P = P^T$, quadratic functions)
- $V(x) = x_1^2 + x_2^2 - 4$ is negative definite in the circle in $\mathbb{R}^2$ with the ray equal to 2.
Fundamental theorem of stability, local asymptotic stability (1)

- We can state the first theorem of Lyapunov in the local case.

**Theorem**

*Let us consider the equilibrium point $x_e = 0$ and a domain $\mathcal{D}$ containing 0. Let $V : \mathcal{D} \rightarrow \mathbb{R}$, be a $C^1$ function such that:

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for } x \in \mathcal{D} - \{0\},$$

$$\dot{V}(x) \leq 0 \text{ in } \mathcal{D},$$

then $x = 0$ is a stable equilibrium point. Moreover, if

$$\dot{V}(x) < 0 \text{ in } \mathcal{D} - \{0\},$$

then $x = 0$ is asymptotically stable.*
Fundamental theorem of stability, local asymptotic stability (2)

- A function \( V \) satisfying (2)-(3) or (2)-(4) is called a Lyapunov function.
- The surface \( V(x) = c \) is called a level (or surface) set of the Lyapunov function.
Examples (1)

- **Example 1.** Consider the following example:

  \[ \dot{x}(t) = ax(t), \ a < 0. \]

  This is a linear first-order system. Let us consider \( V(x(t)) = \frac{1}{2}x^2(t) \). The time-derivative of \( V \) along the trajectories of the system gives:

  \[ \dot{V}(x) = x\dot{x} = ax^2. \]

  Since \( a < 0 \), it follows that \( \dot{V}(x) < 0 \) for \( x \in \mathbb{R} - 0 \), then, the system is asymptotically stable.
Examples (2)

- **Example 2.** Consider the example of the inverse pendulum.

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -a \sin(x_1(t)) - bx_2(t)
\end{align*}
\]

and the function

\[
V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2
\]

▷ Determine \( \mathcal{D} \).
▷ Is \( V(x) \) a Lyapunov function?
▷ Let us consider another function:

\[
V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x^T P x
\]

Is-it a Lyapunov function?
Examples (3)

- **Determine** $\mathcal{D}$. Recall $V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$
  - We have $V(0) = 0$ for $x_1 = 0$ and $x_2 = 0$.
  - $V(x)$ is positive definite ($V(x) > 0$) over the domain $-2\pi < x_1 < 2\pi$ and for any $x_2$.

- **Is** $V(x)$ **a Lyapunov function?**
  - Compute $\dot{V}(x)$:
    
    $$\dot{V}(x) = a \sin(x_1) \dot{x}_1 + x_2 \dot{x}_2
    = a \sin(x_1)x_2 + x_2(-a \sin(x_1) - bx_2)
    = -bx_2^2$$

  - One gets: $\dot{V}(x) \leq 0$, since $\dot{V}(x) = 0$ for $x_2 = 0$ and $\forall x_1$.
  - The function $V$ satisfies (2)-(3) and therefore is a Lyapunov function.
  - The origin is stable.
Examples (4)

Try another function. Consider \( V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x^TPx \) with 
\[
P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.
\]

▷ \( P \) is chosen to be positive definite: \( p_{11} > 0, p_{22} > 0 \) and 
\[
\det(P) = p_{11}p_{22} - p_{12}^2 > 0.
\]
Then \( V(x) \) is positive definite.

▷ Compute \( \dot{V}(x) \):

\[
\dot{V}(x) = a \sin(x_1)\dot{x}_1 + \frac{1}{2}(\dot{x}^TPx + x^TP\dot{x})
\]
\[
= a \sin(x_1)\dot{x}_1 + (x_1p_{11} + x_2p_{12})\dot{x}_1 + (x_1p_{12} + x_2p_{22})\dot{x}_2
\]
\[
= a \sin(x_1)x_2 + (x_1p_{11} + x_2p_{12})x_2
\]
\[
+(x_1p_{12} + x_2p_{22})(-a \sin(x_1) - bx_2)
\]
\[
= a \sin(x_1)x_2(1 - p_{22}) + x_1x_2(p_{11} - bp_{12}) + x_2^2(p_{12} - bp_{22})
\]
\[
- a \sin(x_1)x_1p_{12}
\]

▷ Possible choice for \( P \):

\( 1 - p_{22} = 0 \) then \( p_{22} = 1 \)
\( p_{11} - bp_{12} = 0 \) then \( p_{11} = bp_{12} > 0 \)
\( p_{12} - bp_{22} = p_{12} - b < 0 \) then \( 0 < p_{12} < b \) and one can choose \( p_{12} = \frac{b}{2} \)
Examples (5)

- With such a matrix $P =$

$$
\begin{bmatrix}
\frac{b^2}{2} & \frac{b}{2} \\
\frac{b}{2} & \frac{2}{1}
\end{bmatrix}
$$

- one gets for $\dot{V}(x)$:

$$
\dot{V}(x) = -\frac{b}{2}x_2^2 - \frac{ab}{2}\sin(x_1)x_1
$$

- $V(x)$ is positive definite ($V(x) > 0$) over the domain $-\pi < x_1 < \pi$ and for any $x_2$.

- $V$ satisfies (2)-(4).

- The origin is asymptotically stable.
Estimation of the attraction region (1)

- We have thanks to the notion of Lyapunov function studied the **local stability** of the equilibrium point. In particular to simplify, we have studied the stability of the origin ($x_e = 0$).

  ▷ Recall that to prove this we use the conditions (2)-(4):

    \[
    V(x_e) = 0 \text{ and } V(x) > 0 \text{ for } x \in \mathcal{D} - \{0\}, \\
    \dot{V}(x) < 0 \text{ in } \mathcal{D} - \{0\},
    \]

- By definition, when we consider an asymptotically stable equilibrium point $x_e = 0$, that signifies that it is **stable and attractor**.

- A nature question arises: **Is it possible to estimate the attraction region of the equilibrium point** denoted $B_r$, i.e. the set of initial conditions such that the solutions of (1) initiated at this initial conditions converge toward $x_e = 0$?
Estimation of the attraction region (2)

- Find such a set is very complicated, or even often impossible.
- On the other hand, an estimate of the set may be obtained thanks to the Lyapunov theory.
- Let $V(x)$ a Lyapunov function and define the set
  \[ \Omega_c = \{ x \in \mathcal{D}, V(x) \leq c, c > 0 \}. \]
  such that $V(x) > 0$ for $x \in \Omega_c - \{0\}$ and $\dot{V}(x) < 0$ in $\Omega_c - \{0\}$, then any trajectory initialized inside $\Omega_c$ remains in $\Omega_c$ as
  \[ \dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq c, \forall t \geq 0 \]
  Moreover, \( \lim_{t \to +\infty} x(t) = 0 \) by definition.
  \[ \Omega_c \] is an estimate of the domain of attraction (also called the basin of attraction).
- The idea is to find the largest $c > 0$, but note that this may be very conservative.
Example

Consider the system described

\[
\begin{cases}
    \dot{x}_1(t) = x_1(t)(x_1^2(t) + x_2^2(t) - 2) - 4x_1(t)x_2^2(t) \\
    \dot{x}_2(t) = 4x_1^2(t)x_2(t) + x_2(t)(x_1^2(t) + x_2^2(t) - 2)
\end{cases}
\]

If \( V(x) = x_1^2 + x_2^2 \), we have

\[
\dot{V}(x) = 2(x_1^2 + x_2^2)(x_1^2(t) + x_2^2(t) - 2)
\]

Define the set

\[
\Omega_c = \{ x \in \mathcal{D}, V(x) \leq 2 \}.
\]

\( \dot{V}(x) < 0 \) for \( x \in \Omega_c - \{0\} \). Then the equilibrium \( x_e = 0 \) is asymptotically stable and \( \Omega_c \) is an estimation of the basin of attraction.
How is it possible to define the performance through a Lyapunov function?

We have the following interesting result

**Lemma**

*If* $W(t)$ *is a real function and if there exists a positive real number* $\beta$ *such that*

$$\dot{W}(t) + \beta W(t) \leq 0$$

*Then we have*

$$W(t) \leq W(0)e^{-\beta t}$$

To apply the lemma, the idea is to find a relation between $V(x)$ and $\dot{V}(x)$. The performance is then related to the rate of decay of a Lyapunov function.
Returning to the previous example, we can remark that

$$\dot{V}(x) = 2V(x)(V(x) - 2)$$

Taking $W(t)$ such that

$$W(t) = \frac{V(x(t))}{2 - V(x(t))}$$

We can show that (Show it)

$$\dot{W}(t) + 4W(t) = 0$$

Then

$$W(t) = W(0)e^{-4t} \quad \text{with} \quad W(0) = \frac{V(x(0))}{2 - V(x(0))}$$

We can deduce that

$$V(x(t)) = \|x(t)\|^2 = \frac{2W(0)e^{-4t}}{1 + W(0)e^{-4t}} \leq 2W(0)e^{-4t}$$
Fundamental theorem of stability, global asymptotic stability (1)

- Until now, we have manipulated a theorem of local asymptotic stability.

- A natural question arises: Under which conditions is obtained an attraction domain equal to \( \mathbb{R}^n \)?

**Theorem**

*Let us consider an equilibrium point \( x = 0 \) for system (1). Let \( V : \mathbb{R}^n \to \mathbb{R} \), be a \( C^1 \) function such that:*

\[
\|x(t)\| \to +\infty \Rightarrow V(x) \to +\infty \tag{5}
\]

\[
V(x_e) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \neq 0, \tag{6}
\]

\[
\dot{V}(x) < 0 \quad \forall x \neq 0, \tag{7}
\]

*then \( x = 0 \) is globally asymptotically stable.*
Fundamental theorem of stability, global asymptotic stability (2)

- The condition (5)

\[ \|x(t)\| \to +\infty \Rightarrow V(x) \to +\infty \]

expresses that the function \( V \) is radially unbounded.

- This property is needed to prove the global asymptotic stability.

- Thanks to this property one can prove that the level set \( \{x \in \mathbb{R}^n; V(x) \leq c\} \) is bounded for every \( c > 0 \).

- **Example.** Consider \( V(x) = x_2^2 + \frac{x_1^2}{1+x_1^2} \). For \( c = 5 \) and \( X_2 = 2 \), the set is unbounded in the \( x_1 \)-direction.

![Diagram of level sets](image)
Table of Contents

1. Introduction to stability
2. Stability in the sense of Lyapunov
3. Lyapunov method
4. Linear systems and linearization
5. Time-Varying Systems: the linear case
6. Input to State Stability (ISS)
Linear Systems and Linearization
Stability of linear system (1)

- Let us recall some elements on linear systems

\[ \dot{x} = Ax, \quad A \in \mathbb{R}^{n \times n} \]

- Let us consider the equilibrium point \( x_e = 0 \) of the linear system.
- If \( \det(A) = 0 \), the system has an infinite number of equilibrium points
- In that case, at least one eigenvalue of matrix \( A \) is null
Example

- The classical transfer function for a DC motor is given by

\[
G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s(1 + Ts)}, \quad K > 0, \; T > 0
\]

A state equation is given by (Show it)

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{K}{T} \end{bmatrix} u
\]

The equilibrium points are characterized by

\[
x_e = \begin{bmatrix} x_e1 \\ 0 \end{bmatrix}
\]

(Give a physical interpretation)
Stability of linear system (2)

Theorem

The equilibrium point $x_e = 0$ is:

1. stable if and only if for all $i = 1, \ldots, n$, $\text{Re}(\lambda_i) \leq 0$ and for all pure imaginary eigenvalues, of algebraic multiplicity $q_i \geq 2$, one verifies that $\text{rank}(A - \lambda_i I) = n - q_i$.

2. asymptotically stable if and only if $\text{Re}(\lambda_i) < 0$, $i = 1, \ldots, n$.

3. unstable if and only if there is, at least, one $i$ such that $\text{Re}(\lambda_i) > 0$.

- If the system is asymptotically stable then the matrix $A$ is invertible and there is only one equilibrium point $x_e = 0$.
- The invertibility of matrix $A$ is a necessary but not a sufficient condition for asymptotic stability.
- $\text{rank}(A - \lambda_i I) = n - q_i$ is equivalent to $\text{Ker}(A - \lambda_i I) = q_i$. 
Stability of linear system (3)

- Relation with Lyapunov stability theory
  - Let us consider a quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T$, $P > 0$.
    
    $$\dot{V}(x) = x^T (A^T P + PA)x = -x^T Q x$$

    where
    
    $$A^T P + PA = -Q$$

  - We retrieve the classical Lyapunov equation
  - If for $Q > 0$ there exists a solution $P = P^T > 0$, then the equilibrium point is asymptotically stable.
  - If $A$ is Hurwitz, then $P$ solution of the Lyapunov equation is unique.
  - If the system is asymptotically stable then, $\forall Q = Q^T > 0$, there exists $P = P^T > 0$
    
    $$A^T P + PA = -Q$$

  - This result seems to be surprising because, in general, Lyapunov theory provides sufficient conditions for stability
The previous facts are summarized in the following theorem.

**Theorem**

The following assertions are equivalent:

1. The system \( \dot{x}(t) = Ax(t) + Bu(t) \) is asymptotically stable.
2. For all \( Q = Q^T > 0 \), there exists a matrix \( P = P^T > 0 \) which satisfies the Lyapunov equation:
   \[
   A^TP + PA = -Q
   \]
3. There exists a matrix \( W = W^T > 0 \) which satisfies the Lyapunov inequality:
   \[
   A^TW + WA < 0
   \]
4. There exists a matrix \( S = S^T > 0 \) which satisfies the Lyapunov inequality:
   \[
   AS + SA^T < 0
   \]

What is the difference between ii) and iii), Try to deduce iv) from iii)

Show i) \(\rightarrow\) ii), hint: derive the function \( f(t) = e^{A^T}tQe^{At}, \ Q = Q^T > 0 \)
The positive definite symmetric matrices have a specific role in the context of linear time-invariant systems.

**Theorem**

- i) *All the eigenvalues of a symmetric matrix are real*
- ii) *All the diagonal elements of definite positive symmetric matrix are positive*
- iii) *The inverse of a symmetric matrix is symmetric*
- iv) *A symmetric matrix is diagonalisable*
- v) *The eigenvectors of a symmetric matrix are orthogonal*
Indirect Lyapunov method (1)

- One can use the linear approximation of a nonlinear systems in order to conclude on its stability.
- Then, the use of Taylor expansion at the order 1 can be used to prove the local stability of a nonlinear system.

**Theorem**

Let us consider the system $\dot{x}(t) = f(x(t))$, $x_e = 0$ being an equilibrium point. Let us consider $A = \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_e}$.

- If $\text{Re}(\lambda_i) < 0$ then the equilibrium point $x_e = 0$ is locally asymptotically stable.
- If the exists an eigenvalue $\text{Re}(\lambda) > 0$ then the equilibrium point $x_e = 0$ is unstable.
- If the exists an eigenvalue $\text{Re}(\lambda) = 0$ then one cannot conclude.
Table of Contents

1 Introduction to stability
2 Stability in the sense of Lyapunov
3 Lyapunov method
4 Linear systems and linearization
5 Time-Varying Systems: the linear case
6 Input to State Stability (ISS)
Time-Varying Systems: the linear case
Introduction

Recall the general form of a nonlinear system:

\[ \dot{x}(t) = f(t, x(t), u(t)), \]

(1)

Definition

System (1) is said **stationary** if it does not depend on the time variable \( t \). In the opposite case, the system is said to be **time-varying**.

Definition

System (1) is said **autonomous** if it does not depend on the control variable \( u(t) \). In the opposite case, the system is said to be a **controlled system**.

- It is possible to extend the definition of stability to deal with time-varying systems (quite complex). An important notion in this context is the notion of **uniformity**.
Example

Consider the system
\[ \dot{x}(t) = -\frac{x(t)}{1 + t}, \quad x(t_0) \]

The solution reads (Show it)
\[ x(t) = \frac{1 + t_0}{1 + t} x(t_0) \]

The system is asymptotically stable because \( \lim_{t \to \infty} x(t) = 0 \).

But the convergence depends of initial time \( t_0 \Rightarrow \)

The convergence is not uniform.
We concentrate our attention to time-varying linear systems described by

\[ \dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \]

We defined the transition matrix \( \Phi(t, t_0) \) by

\[ x(t) = \Phi(t, t_0)x(t_0), \quad \Phi(t_0, t_0) = I \]

We have

\[ \dot{x}(t) = \dot{\Phi}(t, t_0)x(t_0) = A(t)x(t) = A(t)\Phi(t, t_0)x(t_0) \]

And then

\[ \dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)x(t_0), \quad \Phi(t_0, t_0) = I \]

It is then possible to compute \( \Phi(t, t_0) \) by numerical integration.
The following results can be proved

**Theorem**

*The system* $\dot{x}(t) = A(t)x(t)$ *is*

i) **uniformly stable if and only if**

$$\| \Phi(t, t_0) \| \leq \gamma, \ \forall t > t_0 \ and \ \gamma > 0$$

ii) **uniformly asymptotically stable if and only if**

$$\| \Phi(t, t_0) \| \leq \gamma e^{-\lambda(t-t_0)}, \ \forall t > t_0 \ and \ \gamma > 0, \ \lambda > 0$$

- It is not necessary to compute the norm of the transition matrix. the inequalities can be checked element by element
- Remark that uniform asymptotic stability is equivalent, in that case, to exponential stability
An intriguing example

- Consider the matrix defined by

\[
A(t) = \begin{bmatrix}
-1 + \frac{3}{2} \cos^2(t) & 1 - \frac{3}{2} \sin(t) \cos(t) \\
-1 - \frac{3}{2} \sin(t) \cos(t) & -1 + \frac{3}{2} \sin^2(t)
\end{bmatrix}
\]

By a simple calculation, we obtain (do it)

\[
\det(\lambda(t)I - A(t)) = \lambda^2(t) + \frac{\lambda(t)}{2} - \frac{1}{2}
\]

The roots are independent of \( t \) and given by

\[
-\frac{1}{4} + \frac{\sqrt{7}}{4}, \quad -\frac{1}{4} - \frac{\sqrt{7}}{4}
\]

- As the real part of eigenvalues is negative, we can expect that the system be stable. In fact this is not the case.
Remark that $A(t + 2k\pi) = A(t)$, $k \in \mathbb{Z}$. The system is said \textbf{periodic}.

Define the following change of coordinate (\textit{Floquet Theory})

$$x(t) = \begin{bmatrix} -\cos(t) & \sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} y(t) = G(t)y(t)$$

Remark that $G(t)$ is bounded (this is important). We have

$$\dot{x}(t) = \dot{G}(t)y(t) + G(t)\dot{y}(t) = A(t)G(t)y(t)$$

and then after some calculations, we obtain

$$\dot{y}(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix} y(t) \Rightarrow y(t) = \begin{bmatrix} y_1(0)e^{t/2} \\ y_2(0)e^{-t} \end{bmatrix}$$
\[ \chi(t) = G(t) \begin{bmatrix} e^{1/2t} & 0 \\ 0 & e^{-t} \end{bmatrix} G^{-1}(t)\chi(0) \]

\[ = \begin{bmatrix} -\cos(t) & \sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} e^{1/2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -\cos(t) & \sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}^{-1} \chi(0) \]

\[ = \begin{bmatrix} e^{-t}\sin^2 t + e^{1/2}t\cos^2 t & \cos t \sin t \left( e^{-t} - e^{1/2}t \right) \\ \cos t \sin t \left( e^{-t} - e^{1/2}t \right) & e^{-t}\left( \cos^2 t + e^{1/2}t\sin^2 t \right) \end{bmatrix} \chi(0) \]

\[ \Phi(t, t_0) \]

- We have \( \lim_{t \to \infty} \|\Phi(t, t_0)\| = \infty \). the system is unstable.

- We cannot extend the results concerning time-invariant linear systems to the case of time-varying linear ones. In particular the notions of pole and zero have to be revisited.
# Table of Contents

1. Introduction to stability
2. Stability in the sense of Lyapunov
3. Lyapunov method
4. Linear systems and linearization
5. Time-Varying Systems: the linear case
6. Input to State Stability (ISS)
<table>
<thead>
<tr>
<th>Introduction to stability</th>
<th>Stability in the sense of Lyapunov</th>
<th>Lyapunov method</th>
<th>Linear systems and linearization</th>
<th>Time-Varying Systems: the case of linear systems</th>
</tr>
</thead>
</table>

**Input to State Stability (ISS)**
Introduction

Consider a nonlinear system of the form:

\[ \dot{x}(t) = f(t, x(t), u(t)), \]  

- \( f \) is a piecewise continuous function with respect to time, locally Lipschitz with respect to \( x \) and \( u \).
- The input \( u(t) \) is piecewise continuous and is a bounded function of time.

One assume that the unforced system

\[ \dot{x}(t) = f(t, x(t), 0) \]

possesses an equilibrium point globally asymptotically stable in \( 0 \).

A natural question arises: What can be said about the behavior of (8) when it is subject to a bounded disturbance input \( u(t) \)?
The case of linear systems (1)

- First, let us see what can be the answer in the case of a linear system.

- The solution of the state equation for the system
  \[ \dot{x}(t) = Ax(t) + Bu(t), \quad A \text{ is supposed Hurwitz} \]

  reads:
  \[ x(t) = e^{At}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau. \]

- Moreover since \( A \) is Hurwitz there exists \( k, \lambda > 0 \), such that \( \|e^{At}\| \leq ke^{-\lambda t} \).
  Then, one obtains:
  \[ \|x(t)\| \leq ke^{-\lambda t}\|x(0)\| + \int_0^t ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau. \]

  \[ \leq ke^{-\lambda t}\|x(0)\| + \frac{k\|B\|}{\lambda} \sup_{0 \leq \tau \leq t} \|u(\tau)\| \]
The case of linear systems (2)

- Then, one can deduce
  - Bounded input $\Rightarrow$ bounded states.
  - The bound on the states is proportional to the bound on the input.

- Is it the same for nonlinear systems?
A nonlinear example

Consider the following example:

\[ \dot{x}(t) = -x(t) + (x^2(t) + 1)u(t) \]

- Without perturbation input, the equilibrium point 0 is GAS (Globally asymptotically stable). (Explain why)

- Let us consider \( u(t) = (2t + 2)^{-1/2} \), \( \lim_{t \to \infty} u(t) = 0 \). Even though, the resulting system is unstable. (Show it in details)

- Let us consider \( u(t) = 1/2 \). Then the resulting system becomes unstable. (Show it in details)
Definition of class $\mathcal{K}$, $\mathcal{KL}$, $\mathcal{L}$ functions (1)

- In order to discuss about Lyapunov stability and ISS, it is useful to manipulate comparison functions.

**Definition**

A continuous function $\alpha$ of $[0, a]$ valued in $[0, +\infty]$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r \to +\infty} \alpha(r) = +\infty$.

**Example 1.** $\alpha(x) = \tan^{-1}(x)$ is strictly increasing since one gets $\frac{\partial \alpha}{\partial x} = \frac{1}{1+x^2} > 0$. It belongs to class $\mathcal{K}$ but not to class $\mathcal{K}_\infty$ because $\lim_{x \to \infty} \alpha = \frac{\pi}{2} < \infty$.

**Example 2.** $\alpha(x) = x^c$, $c > 1$, is strictly increasing since one gets $\frac{\partial \alpha}{\partial x} = cx^{c-1} > 0$. Furthermore, $\lim_{x \to \infty} \alpha = \infty$, thus it belongs to class $\mathcal{K}_\infty$. 
Definition of class $\mathcal{K}$, $\mathcal{KL}$, $\mathcal{L}$ functions (2)

**Definition**

A continuous function $\phi$ of $[0, \infty]$ valued in $[0, +\infty]$ is said to be of class $\mathcal{L}$ if it is strictly decreasing and $\lim_{s \to +\infty} \phi(s) = 0$.

**Definition**

A two-arguments function is said to be of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ with respect to the first argument and of class $\mathcal{L}$ with respect to the second one.
Definition of class $\mathcal{K}$, $\mathcal{KL}$, $\mathcal{L}$ functions (3)

- **Example 1.** Consider a function $\beta(x, y) = \frac{x}{kxy + 1}$, $k > 0$.
  - It is strictly increasing in $x$ since we have: $\frac{\partial \beta}{\partial x} = \frac{1}{(kxy + 1)^2} > 0$;
  - It is strictly decreasing in $y$ since we have $\frac{\partial \beta}{\partial y} = \frac{-kx^2}{(kxy + 1)^2} < 0$;
  - Furthermore, $\lim_{y \to \infty} \beta(x, y) = 0$.
  - It is a function of class $\mathcal{KL}$.

- **Example 2.** Is the function $\beta(x, y) = x^c e^{-\alpha y}$, $\alpha > 0$, $c > 1$, of class $\mathcal{KL}$?
Lyapunov theorem using comparison functions

- Consider a dynamical system described by the differential equation (1) and an equilibrium point $x_e = 0$.

Theorem

Consider the equilibrium point $x_e = [0 \ 0]^T$ and a domain $\mathcal{D}$ involving 0. Let $V(x, t) : \mathcal{D} \times \mathbb{R}^+ \to \mathbb{R}$, a $C^1$ function such that:

\[ \alpha_1(\|x\|) \leq V(x, t) \leq \alpha_2(\|x\|) \quad (9) \]
\[ \dot{V}(x, t) \leq -\alpha_3(\|x\|), \quad (10) \]

then, system (1) is

- Stable if $\alpha_1, \alpha_2$ are class $\mathcal{K}$ functions and $\alpha_3 \geq 0$ on $\mathcal{D}$.
- Asymptotically stable if $\alpha_1, \alpha_2, \alpha_3$ are class $\mathcal{K}_\infty$ functions.
Definition of the Input to State Stability (ISS)

Definition

A system of the form \( \dot{x}(t) = f(x(t), u(t)) \) is said to be Input to State Stable if and only if it exists a function \( \beta \) of class \( \mathcal{KL} \) and a function \( \gamma \) of class \( \mathcal{K} \) such that for all the initial conditions \( x_0 \) and all the bounded inputs \( u(t) \), the solution \( x(t) \) exists for \( t \geq 0 \) and satisfies:

\[
\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\sup_{0 \leq \tau \leq t} \|u(\tau)\|).
\]

- If \( u = 0 \), then this definition turns up to the global asymptotic stability of the origin.
Properties

- Let us give some properties related to ISS
  - ISS implies the Bounded Input Bounded State (BIBS);
  - \( \lim_{t \to \infty} u(t) \) implies \( \lim_{t \to \infty} x(t) \);
  - The origin of \( \dot{x} = f(x, 0) \) is GAS.

- For a linear system, GAS implies ISS, (Show it)

- Be careful, this is not true for a nonlinear system.
GAS does not imply ISS

- **Example.** Consider the system

\[
\dot{x} = u - \text{sat}(x) = u - \begin{cases} 
1 & \text{if } x > 1 \\
 x & \text{if } -1 \leq x \leq 1 \\
-1 & \text{if } x < -1
\end{cases}
\]

- The system is GAS for \( u = 0 \). (*Show it*)

- For \( u = 2 \) and \( x(0) = 2 \), the solution of the system is \( x(t) = 2 + t \), which is not bounded (even if \( u \) is bounded). (*Show it*)
Lyapunov theorem for the ISS

Now we present a result using a Lyapunov function to prove ISS for a nonlinear system.

**Theorem**

Let us consider a function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, a $C^1$ function such that:

$$
\alpha_1(\|x\|) \leq V \leq \alpha_2(\|x\|)
$$

$$
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x(t), u) \leq -\alpha_3(x), \quad \forall \|x\| \geq \rho(\|u\|) > 0,
$$

where $\alpha_1, \alpha_2$ are class $\mathcal{K}_\infty$ functions, $\rho$ is a class $\mathcal{K}$ function and $\alpha_3$ is a positive definite function on $\mathbb{R}^n$, then the system (8) is ISS.
Examples (1)

- **Example 1.** Consider the following system:

\[ \dot{x}(t) = -x^3 + u, \]

- The origin of the unforced system \((u = 0)\) is globally asymptotically stable. *(Show it)*

- Take the Lyapunov function \(V(x) = \frac{1}{2}x^2\). Then one gets its time-derivative:

\[ \dot{V}(x) = x\dot{x} = x(-x^3 + u) = -x^4 + xu \]

- We can add without changing anything (with \(0 < \theta < 1\)):

\[
\begin{align*}
\dot{V}(x) &= -x^4 + xu = -x^4 + xu + \theta x^4 - \theta x^4 = -(1 - \theta)x^4 + xu - \theta x^4 \\
&= -(1 - \theta)x^4 + x(u - \theta x^3)
\end{align*}
\]

- Then one obtains \(\dot{V}(x) \leq -(1 - \theta)x^4\) provided that \(x(u - \theta x^3) < 0\), or equivalently, for any \(x\) such that \(|x| > \left(\frac{|u|}{\theta}\right)^{\frac{1}{3}}\).

- ISS is verified with \(\rho(||u||) = \left(\frac{|u|}{\theta}\right)^{\frac{1}{3}}\).
Examples (2)

**Example 2.** Consider the following system:

\[ \dot{x}(t) = -x - 2x^3 + (1 + x^2)u^2, \]

▷ The origin of the unforced system \((u = 0)\) is globally exponentially stable.

▷ Take the Lyapunov function \(V(x) = \frac{1}{2}x^2\). Then one gets its time-derivative:

\[ \dot{V}(x) = x\dot{x} = x(-x - 2x^3 + (1 + x^2)u^2) = -x^2 - 2x^4 + x(1 + x^2)u^2 \]

▷ One can still write:

\[ \dot{V}(x) = -x^4 - x^4 - x^2 + x(1 + x^2)u^2 = -x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \]
\[ = -x^4 + (1 + x^2)(-x^2 + xu^2) = -x^4 + (1 + x^2)x(-x + u^2) \]

▷ Then one obtains \(\dot{V}(x) \leq -x^4\) provided that \(-x^2 + xu^2 < 0\), or equivalently, for any \(x\) such that \(|x| > u^2\).

▷ ISS is verified with \(\rho(||u||) = u^2\).
ISS for cascade systems

Now we present a result for two systems in cascade.

**Theorem**

If the systems $\dot{x}_1 = f_1(x_1, x_2)$ and $\dot{x}_2 = f_2(x_2, u)$ are input-to-state stable, then the cascade connection

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_2, u)
\end{align*}
\]

is input-to-state. Consequently if $\dot{x}_1 = f_1(x_1, x_2)$ is ISS and the origin of $\dot{x}_2 = f_2(x_2, 0)$ is globally asymptotically stable, then the origin of the cascade connection

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_2, 0)
\end{align*}
\]

is globally asymptotically stable.
Example (1)

- **Example.** Consider the following system:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2^2 = f_1(x_1, x_2) \\
\dot{x}_2 &= -x_2 + u = f_2(x_2, u)
\end{align*}
\]

- Let us first verify that the first system is ISS (with respect to \(x_2\)).
  - By considering \(V(x_1) = \frac{1}{2}x_1^2\). Then one gets its time-derivative:
    \[
    \dot{V}(x_1) = x_1 \dot{x}_1 = x_1(-x_1 + x_2^2) = -x_1^2 + x_1x_2^2
    \]
  - We can add without changing anything (with \(0 < \theta < 1\)):
    \[
    \dot{V}(x_1) = -x_1^2 + x_1x_2^2 = -x_1^2 + x_1x_2^2 + \theta x^2 - \theta x^2 = -(1 - \theta)x^2 + x_1(x_2^2 - \theta x_1)
    \]
  - Then one obtains \(\dot{V}(x_1) \leq -(1 - \theta)x_1^2\) provided that \(x_1(x_2^2 - \theta x_1) < 0\), or equivalently, for \(|x_1| > \frac{x_2^2}{\theta}\).
  - ISS is verified with \(\rho(\|x_2\|) = \frac{x_2^2}{\theta}\).
Example (2)

- Now we can verify that the second system is ISS by using $V(x_2) = \frac{1}{2}x_2^2$ and by using similar arguments as previously.

- Furthermore, $\dot{x}_2 = f_2(x_2, 0) = -x_2$ is globally asymptotically stable.
  - Then, one can conclude by evoking the theorem that the cascade connection is ISS.
To go further

- One can study several extension of the ISS property:
  - Regional (or local) ISS. One considers that the initial condition and the exogenous signal are bounded in some sets.
  - iISS (integral ISS). integral ISS which corresponds to have a condition using the integral of the norm of the exogenous signal.
  - IOS. For system with output, we bound the norm of the output instead of the norm of the state.
  - IOSS. We bound the state in function of the norm of the state, the output and the exogenous signal.