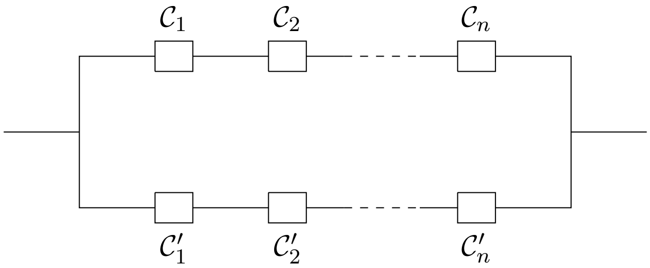
Worksheet 1 - Exponential distribution

Exercise 1. Reliability and failure rate. A system is composed of n components $\mathcal{C}_i,\ 1 \leq i \leq n$. Denote X_i the time to failure of $\mathcal{C}_i$, and assume that $X_1, \ldots, X_n$ are independent. Recall that the reliability function R_i of component $\mathcal{C}_i$ and its failure rate λ_i are defined for all $t > 0$ by

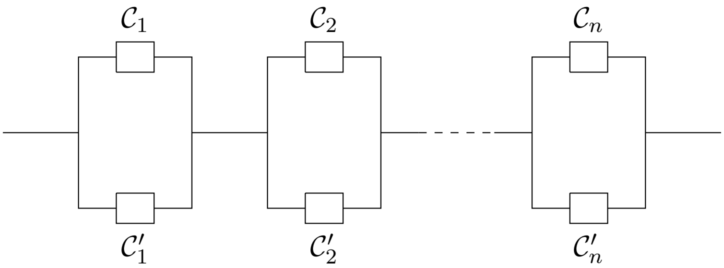
$$R_i(t) = \mathbb{P}\left(X_i > t\right) \qquad \text{and} \qquad \lambda_i(t) = \frac{f_i(t)}{R_i(t)},$$

where f_i denotes the density function of X_i.

1. First, we put these components in a series system and we denote T^* the lifetime of the system.
   (a) Compute the reliability R^* and the failure rate λ^* of this series system.
   (b) What happens in the case of exponential lifetimes? Compute $\mathbb{E}\left[T^*\right]$ in this case.
2. Then, we put these components in a parallel system and denote $\tilde{T}$ the lifetime of this new system.
   (a) Compute the reliability $\tilde{R}$ of this parallel system.
   (b) What is the reliability in the case of i.i.d. exponential lifetimes? Deduce $\mathbb{E}\left[\tilde{T}\right]$ in this case.
3. We aim at increasing the reliability of a system composed of n components in series. To do so, we get independent copies of each component (denoted $\mathcal{C}_i'$) and decide to compare two systems.
   - System 1: the entire series systems of respectively $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and $\mathcal{C}_1', \ldots, \mathcal{C}_n'$ are put in parallel.



   - System 2: the parallel systems of each component with his copy are put in series.



   (a) Express the reliability function of each system in terms of the R_i's.
   (b) Prove that in the case of i.i.d. exponential lifetimes with failure rate $\lambda > 0$, this comes down to comparing $\left(1 - \left(1 - e^{-\lambda t}\right)^2\right)^n$ and $1 - \left(1 - e^{-n\lambda t}\right)^2$. Which system seems more reliable to you?

Exercise 2. The Gamma distribution. Let $\alpha, \lambda > 0$ and consider $X \sim \Gamma(\alpha, \lambda)$.

1. Compute the Laplace transform of X.
2. Deduce $\mathbb{E}\left[X\right]$ and $\mathrm{Var}(X)$.

3. Let $\beta > 0$ and consider $Y \sim \Gamma(\beta, \lambda)$ independent of X. Show that $X + Y \sim \Gamma(\alpha + \beta, \lambda)$.
4. Deduce the distribution of the sum of n i.i.d. exponential random variables with parameter λ.

Exercise 3. Memoryless property and Defective cellphones.

(A) Generalization of the memoryless property.

Let X be an exponentially distributed r.v. $X \sim \mathcal{E}(\lambda)$, with parameter $\lambda > 0$, and U be a continuous real-valued random variable with density g. Assume $X \perp\!\!\!\perp U$. Prove that for all $t, s > 0$,

$$\mathbb{P}\left(X + U > t + s | X > t\right) = \mathbb{P}\left(X + U > s\right).$$

(B) Cellphone warranty. A cellphone manufacturer offers warranty for 8 months with a free replacement if the device is defective. The manufacturer assumes that each unit has a lifetime that can be modeled by an exponential r.v. with mean equal to 24 months.

1. What is the (theoretical) proportion of defective devices that will be replaced for free?
2. What warranty should the manufacturer offer in order to replace only 10% of the devices for free?
3. What is the density function for duration of devices that do not have any failure before 8 months?
4. Charles is a VIP member, he has infinite warranty. He can replace his phone every time it fails. The manufacturer would like to know the probability that 3 phones will not cover 5 years if Charles' first cellphone lasts more than a year. Compute this conditional probability assuming the lifetimes of Charles' cellphones are i.i.d. exponential r.v.