Worksheet 2 - Homogeneous Poisson process

Exercise 1. Defective cellphones (point process approach).
Consider Part **(B) "Cellphone warranty"** of Exercise 3, Worksheet 1.
A cellphone manufacturer offers warranty for 8 months with a free replacement if the device is defective. The manufacturer assumes that each unit has a lifetime that can be modeled by an exponential variable with mean equal to 24 months.

Recall that Charles is a VIP member, he has infinite warranty. He can replace his phone every time it fails. The manufacturer would like to know the probability that 3 phones will not cover 5 years if Charles' first cellphone lasts more than a year. Compute this conditional probability assuming the number of times Charles replaces his cellphone can be modeled by a homogeneous Poisson process.

Exercise 2. The inspection paradox (or *Le paradoxe de l'autobus* in French).

You have an infinite stock of light bulbs with lifetimes which can be modeled by i.i.d. random variables with exponential distribution with parameter $\lambda > 0$. At time $t = 0$, we switch on the light and we change the light bulb as soon as it burns out. The times we change the light bulbs

$$0 < T_1 < T_2 < \ldots < T_n < \ldots$$

thus define a Poisson process $(N_t)_{t\geq 0}$ with rate λ. Denote N_t the number of consumed light bulbs by time t. We decide to approximate the parameter λ by the following "inspection" method:

- Fix a time $t > 0$.
- Observe the lifetime of the "inspected" light bulb at time t, that is $T_{N_t+1} - T_{N_t}$.

We denote $X_t = T_{N_t+1} - t$ the residual lifetime of the inspected bulb at time t, and $Y_t = t - T_{N_t}$ the "age" of the inspected bulb at time t.

1. Prove that $X_t \sim \mathcal{E}(\lambda)$.
2. Compute the survival function of Y_t. Is it exponentially distributed?
3. Prove that X_t and Y_t are independent random variables.
4. Compute the average lifetime of the inspected bulb at time t.
5. Compare it to the average lifetime of the first bulb.

We can see that, for all time $t > 0$, the expected lifetime of the inspected bulb is greater than the average lifetime of the non-inspected bulbs. This phenomenon is often referred to the inspection paradox.

Exercise 3. Small/Large claims. The time unit considered here is a week. A small insurance company has in average one claim per day. It classifies each claim in two categories;

- *small claim* if the claim is less than or equal to 5000 €,
- *large claim* if the claim is greater than 5000 €.

It estimates that large claims occur in 10% of the cases. We model the claim occurrences by a homogeneous Poisson process N with rate $\lambda > 0$, and denote N^I (resp. N^{II}) the point process counting the small claims (resp. large claims).

1. Determine the rate λ.
2. What is the probability of at least 1 large claim in a week?
3. Knowing there has been 5 large claims in 2 weeks, what is the expected number of claims in that interval?
4. There has been 30 accidents in February. What is the probability that 3 of them where large claims?