Chapter 2

Homogeneous Poisson processes

I Definitions

Before defining homogeneous Poisson processes, let us now introduce the notion of point processes on $\mathbb{R}_+$. We observe the occurrences times of a phenomenon, such as failures in reliability theory, or claims in actuarial sciences.

I.1 Point processes and counting processes

There are several ways of defining a point process depending which quantity is considered.

A) Counting process

The first perspective consists in counting the number of events occurring in any time interval (or time window).

Definition 2.1 (Counting process).
A stochastic process $N = (N_t)_{t\in\mathbb{R}_+}$ is said to be a *counting process* if, for all $t \geq 0$, N_t represents the total number of "events" that occur by time t.

Example 2.2.
Here are some examples of counting processes.

a) In reliability theory, one may denote N_t the number of failures of a reparable component or before time t. An event thus corresponds to a failure.

b) In actuarial science, N_t may represent the number of accidents an insurance company has to cover by time t. Here an event is a claim.

c) If we say an event occurs whenever a child is born, then N_t equals the total number of people who where born by time t.

d) Let N_t count the number of goals scored by the French team during the Football World Cup final after t minutes of game. An event if this process occurs when a player scores a goal.

Property 2.3.
A counting process satisfies the following properties.

(i) For all $t \geq 0$, $N_t \geq 0$.

(ii) $t \mapsto N_t$ is non-decreasing, that is for all $s \leq t$, $N_s \leq N_t$.

(iii) $t \mapsto N_t$ is piecewise constant, with values in $\mathbb{N}$.

(iv) $(N_t)_{t \in \mathbb{R}_+}$ is càdlàg, that is right-continuous and left-limited (for *continue à droite, limitée à gauche* in French).

(v) For all $0 \leq s < t$, $N_t - N_s$ counts the number of events that occur in the interval $(s, t]$.

In this lecture, we restrict attention to counting processes N whose realizations are *locally finite* (i.e. the number of events in any bounded domain in $\mathbb{R}_+$ is finite). In particular, this is true as soon as for all $t \geq 0$, N_t is almost surely finite.

B) Point process

The second perspective consists in considering the times each "event" occurs.

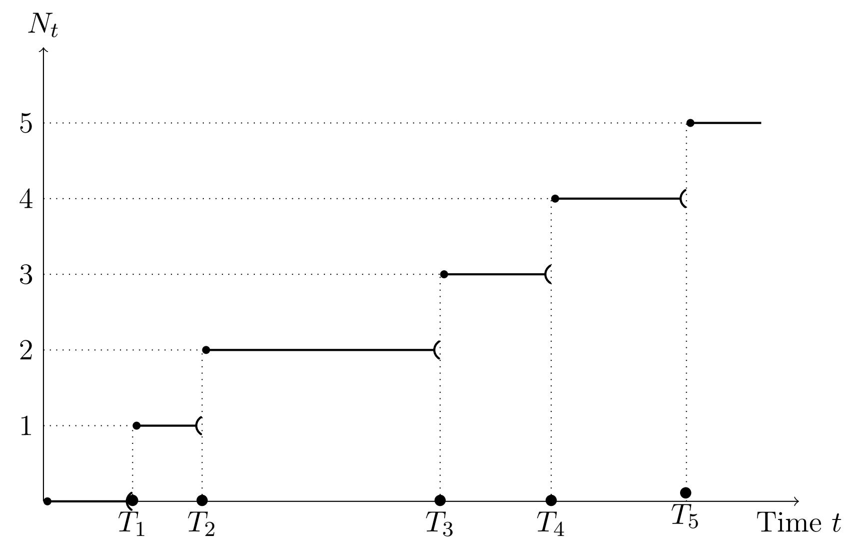
Definition 2.4 (Point process).
A point process on $\mathbb{R}_+$ is a countable subset of $\mathbb{R}_+$, each point representing the time occurence of an "event". It may be formally defined as the intersection between $\mathbb{R}_+$ and the non-decreasing sequence of times the event occurs, also called *arrival times*, and denoted

$$0 \leq T_1 \leq T_2 \leq \ldots \leq T_n \leq \ldots$$

where the T_n belong to $\mathbb{R}_+ \cup \{+\infty\}$.

The knowledge of the counting process N is equivalent to the knowledge of the arrival times

$$0 \leq T_1 \leq T_2 \leq \ldots \leq T_n \leq \ldots$$



Indeed, on the one hand, if we know the counting process, it is possible to recover the arrival times as

$$T_n = \inf\{t \geq 0 \ ; \ N_t \geq n\}, \tag{2.1}$$

with the convention that $\inf\{\varnothing\} = +\infty$. On the other hand, if we know the point process, it is possible to define the corresponding counting process for all $t \geq 0$ by

$$N_t = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}}. \tag{2.2}$$

Notice that this sum converges as soon as the point process does not have any accumulation point, that is the process is locally finite.

Proposition 2.5 (Link between counting and point processes).

Let N be a counting process and denote $(T_n)_{n\in\mathbb{N}^*}$ the corresponding arrival times. Then, for all n in $\mathbb{N}^*$ and all $t \geq 0$,

(i) $\{T_n \leq t\} = \{N_t \geq n\}$.

(ii) $\{T_n > t\} = \{N_t < n\}$.

(iii) $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$.

More generally, for all n in $\mathbb{N}^*$ and all $(t_1, t_2, \ldots, t_n)$ in $(\mathbb{R}_+)^n$,

$$\{T_1 \leq t_1, T_2 \leq t_2, \ldots, T_n \leq t_n\} = \{N_{t_1} \geq 1, N_{t_2} \geq 2, \ldots, N_{t_n} \geq n\}.$$

It is thus equivalent to define the distribution of the point process by the one of $(T_n)_{n\geq 1}$ or the one of $(N_t)_{t\in\mathbb{R}_+}$.

From now on, unless specified, N refers to both the counting process $(N_t)_{t\in\mathbb{R}_+}$ and its associated point process $(T_n)_{n\in\mathbb{N}^*}$, and we do not distinguish between "counting" and "point" processes.

C) Some basic concepts

In this lecture, we only consider *simple* point processes, for which several events cannot occur at the same time.

- On the one hand, from a "point process" perspective, this means that the random variables (T_n) are a.s. pairwise distinct on $\mathbb{R}_+$. More precisely, for all n in $\mathbb{N}^*$ and almost all ω in Ω,

$$T_n(\omega) < +\infty \quad \Longrightarrow \quad T_n(\omega) < T_{n+1}(\omega).$$

  Essentially, it means that for almost all $\omega \in \Omega$, the sequence $(T_n(\omega))_{n\in\mathbb{N}^*}$ is strictly increasing as long as the $T_n(\omega)$ are finite.

- On the other hand, from a "counting process" perspective, it means that all the jumps of the counting process $(N_t)_{t\in\mathbb{R}_+}$ are equal to 1.

Definition 2.6 (Regular point process).

A point process N is said to be *regular* if for all $t \geq 0$,

$$\lim_{h\to 0} \frac{\mathbb{P}(N_{t+h} - N_t \geq 2)}{h} = 0.$$

For instance in reliability theory, it means that the system will not experience two or more failures simultaneously. In particular, a regular point process is simple.

Definition 2.7 (Mean function and rate of a counting process).

Let N be a counting process.

- The **mean function** m of N is defined by

$$m : t \in \mathbb{R}_+ \mapsto m(t) = \mathbb{E}[N_t] \, .$$

- The **rate** w of N is defined by

$$w : t \in \mathbb{R}_+ \mapsto w(t) = m'(t) = \lim_{h \to 0} \frac{\mathbb{E}[N_{t+h} - N_t]}{h} .$$

Comment 2.8.

i. Note that the mean function and the rate of a counting process are deterministic.

ii. Moreover, even if the counting process is piecewise constant (it has "jumps"), its mean function is not necessarily discontinuous.

iii. If N is a regular point process, the probability of two or more events in $]t, t + h]$ is negligible when h is small. Thus, we may assume that, for small h, $N_{t+h} - N_t$ takes values in $\{0, 1\}$, and that

$$\mathbb{E}[N_{t+h} - N_t] \approx \mathbb{P}(N_{t+h} - N_t = 1) \, .$$

Hence, rate of the process is approximately equal to

$$w(t) \approx \frac{\mathbb{P}(N_{t+h} - N_t = 1)}{h} .$$

Hence, we can thing of $w(t)dt$ as the probability of an occurence "just after time t".

iv. In reliability theory, the rate is called **Rate of Occurence of Failure** (RoCoF). It represents the mean number of failures per unit of time.

v. The rate can be generalized to the *conditional rate*, in the case where we work conditionally on all the history of the process. Unlike the rate, the conditional rate is usually stochastic. In reliability theory, it is called **Failure intensity**.

D) Increments of a point process

Definition 2.9 (Stationary increments).

A point process N is said to have stationary increments if the distribution of the number of points in any interval of time depends only on the length of the interval. In other words, for any $s \geq 0$, the number of points in the interval $(t, t + s]$, that is $N_{t+s} - N_t$ has the same distribution for all t.

Example (2.2 continued).

Recall the examples enumerated above.

a) In reliability theory, the stationarity seams reasonable if we consider that there are no running-in period or aging effect, and if after a repair, the component is "as new" (this case is often referred to as **perfect repair**).

b) In actuarial science, it is a reasonable assumption if, for instance, there are no periods in the year where the accidents seem more likely to happen.

c) If we believe that the earth's population is basically constant (a belief not held at present by most scientists), then the assumption of stationary increments might be reasonable.

d) Not reasonable if we take into account the tiredness of the players.

Definition 2.10 (Independent increments).

A point process N is said to have independent increments if the number of points belonging to disjoint intervals are independent. In other words, for all n in $\mathbb{N}^*$, for all $0 < t_1 < t_2 < \ldots < t_n$, the random variables $N_{t_1} - N_0$, $N_{t_2} - N_{t_1}$, ..., $N_{t_n} - N_{t_{n-1}}$ are independent.

Example (2.2 continued).

Recall the examples enumerated above.

a) In reliability theory, it seams reasonable to assume that the increments are independent if we can assume that a failure does not have repercussions (e.g. **perfect repair** case).

b) It depends on the kind of insurance: in life insurance, the increments can be assumed as independent (the number of deaths at time t does not impact the other customers). Yet, in non-life insurance, the independence assumption for the increments is less realistic. For instance, if an earthquake happens, then there will probably be aftershocks and thus other damages.

c) No. If, for instance, N_t is very large, then it is probable that there are many people alive at time t; this would lead us to believe that the number of new births between time t and time $t + s$ would also tend to be large. Hence it does not seem reasonable to assume that N_t is independent of $N_{t+s} - N_t$.

d) Could be justified if we assume the players' chances of scoring a goal do not depend on how the game has been going (unlikely).

I.2 Homogeneous Poisson process on $\mathbb{R}_+$

The most elementary point process is the homogeneous Poisson point process which is defined as follows.

Definition 2.11.

A point process N is said to be a homogeneous Poisson process with rate $\lambda > 0$ if

(1) $N_0 = 0$.

(2) The process has independent increments.

(3) The number of points in any interval of length $t > 0$ has a Poisson distribution with parameter λt, that is, for all $s \geq 0$,

$$N_{s+t} - N_s \sim \mathcal{P}(\lambda t).$$

The parameter λ is also called the intensity of the Poisson point process.

Property 2.12.

Let N be a homogeneous Poisson process with intensity $\lambda > 0$.

1. N has stationary increments.

2. The mean function of N equals

$$\forall t \geq 0, \quad m(t) = \mathbb{E}[N_t] = \lambda t.$$

In particular, the rate (see Definition 2.7) of the Poisson process N is constant (equal, as expected, to λ):

$$\forall t \geq 0, \quad w(t) = \lambda.$$

3. One has as $t \to 0$,

   (a) $\mathbb{P}(N_t = 0) = e^{-\lambda t} = 1 - \lambda t + o(t)$.

   (b) $\mathbb{P}(N_t = 1) = \lambda t e^{-\lambda t} = \lambda t + o(t)$.

   (c) $\mathbb{P}(N_t \geq 2) = \sum_{k=2}^{+\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = o(t)$.

Proof. Immediate by Definition 2.11, and since $N_t \sim \mathcal{P}(\lambda t)$. □

Comment 2.13. In particular, $\forall t \geq 0$,

$$\frac{\mathbb{P}(N_{t+h} - N_t \geq 2)}{h} = \frac{\mathbb{P}(N_h \geq 2)}{h} = o(1) \underset{h \to 0}{\longrightarrow} 0.$$

Hence a homogeneous Poisson process is regular (and thus simple).

Definition 2.14.

The point process N is said to be a homogeneous Poisson process with rate (or with intensity) $\lambda > 0$ if

(A) $N_0 = 0$.

(B) The process has stationary and independent increments.

(C) For all $t > 0$, as $h \to 0$,

   (a) $\mathbb{P}(N_t = 1) = \lambda t + o(t)$.

   (b) $\mathbb{P}(N_t \geq 2) = o(t)$.

Proposition 2.15.

Definition 2.11 and Definition 2.14 are equivalent.

Proof. In exercise*.

[⇒] Done in Property 2.12.

[⇐] Assume Definition 2.14. Then, the points (1) and (2) in Definition 2.11 are immediate from (A) and (B) in Definition 2.14.
Remains to prove (3). To do so, let us compute the moment-generating function of N_t for all $t > 0$.

a) Let $h > 0$. Using Assumption (C), prove that for all $u \leq 0$,

$$\mathbb{E}\left[e^{uN_h}\right] = 1 + \lambda h(e^u - 1) + o(h).$$

Solution.

From (C), we obtain

$$\mathbb{P}(N_h = 0) = 1 - [\mathbb{P}(N_h = 1) + \mathbb{P}(N_h \geq 2)] = 1 - \lambda h + o(h).$$

Hence, since $u \leq 0$,

$$\begin{aligned}\mathbb{E}\left[e^{uN_h}\right] &= \underbrace{\mathbb{P}(N_h = 0)}_{1-\lambda h+o(h)} + e^u \underbrace{\mathbb{P}(N_h = 1)}_{\lambda h+o(h)} + \underbrace{\sum_{k\geq 2} e^{uk}\mathbb{P}(N_h = k)}_{\leq \sum_{k\geq 2}\mathbb{P}(N_h=k)=\mathbb{P}(N_h\geq 2)=o(h)} \\ &= 1 + \lambda h(e^u - 1) + o(h).\end{aligned}$$

b) Fix $u \leq 0$ and denote

$$g : t \mapsto \mathbb{E}\left[e^{uN_t}\right].$$

We now aim at proving that for any small h,

$$g(t + h) = g(t)\,[1 + \lambda h(e^u - 1) + o(h)]\,. \tag{2.3}$$

i) Let $h > 0$. Using both properties in Assumption (B), prove that

$$g(t + h) = \mathbb{E}\left[e^{uN_h}\right] g(t).$$

and deduce (2.3). *Hint:* $N_{t+h} = (N_{t+h} - N_t) + N_t$.

Solution.

By stationarity of the increments,

$$(N_{t+h} - N_t) \text{ has the same distribution than } N_h \qquad (\star),$$

and by independence of the increments, since $(0;t] \cap (t;t+h] = \varnothing$,

$$(N_{t+h} - N_t) \quad \perp\!\!\!\perp \quad N_t - N_0 = N_t \qquad (\dagger).$$

Therefore,

$$\begin{aligned}g(t+h) = \mathbb{E}\left[e^{uN_{t+h}}\right] &= \mathbb{E}\left[e^{u(N_{t+h}-N_t)}e^{uN_t}\right] \\ &\overset{(\dagger)}{=} \mathbb{E}\left[e^{u(N_{t+h}-N_t)}\right]\mathbb{E}\left[e^{uN_t}\right] \\ &\overset{(\star)}{=} \mathbb{E}\left[e^{uN_h}\right] g(t).\end{aligned}$$

Equation (2.3) is then immediate from a).

ii) Let $h < 0$. In the same way, prove that

$$g(t) = \mathbb{E}\left[e^{uN_{(-h)}}\right] g(t + h),$$

and deduce (2.3).

Solution.

As in the previous case, since $N_t = (N_t - N_{t+h}) + N_{t+h}$, by stationarity and independence (since $(0; t+h] \cap (t+h; t] = \varnothing$) of the increments, we obtain

$$\begin{aligned} g(t) = \mathbb{E}\left[e^{uN_t}\right] &= \mathbb{E}\left[e^{u(N_t - N_{t+h})} e^{uN_{t+h}}\right] \\ &= \mathbb{E}\left[e^{u(N_t - N_{t+h})}\right] \mathbb{E}\left[e^{uN_{t+h}}\right] \qquad \text{(independence)} \\ &= \mathbb{E}\left[e^{uN_{(-h)}}\right] g(t+h) \qquad \text{(stationarity).} \end{aligned}$$

We deduce from a) that $g(t) = g(t+h)\,[1 - \lambda h(e^u - 1) + o(h)]$, and thus

$$g(t+h) = \frac{g(t)}{[1 - \lambda h(e^u - 1) + o(h)]} = g(t)\,[1 + \lambda h(e^u - 1) + o(h)]\,.$$

c) i. Deduce from (2.3) that g is differentiable at point t, and satisfies the homogeneous linear differential equation with initial condition $g(0)$ (to determine)

$$\begin{cases} g(0) = \dots \\ g'(t) = g(t)\lambda(e^u - 1). \end{cases} \tag{2.4}$$

Solution.

From (2.3), we directly deduce that

$$\frac{g(t+h) - g(t)}{h} = g(t)\,[\lambda(e^u - 1) + o(1)]$$

has a finite limit when $h \to 0$, which equals

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = g(t)\,[\lambda(e^u - 1)]\,.$$

Moreover, $g(0) = \mathbb{E}\left[e^0\right] = 1$.

ii. Solve the differential equation and deduce that

$$g(t) = e^{\lambda t(e^u - 1)}.$$

Solution.

Hence g is the unique solution of the homogeneous linear differential equation with initial condition (2.4), that is

$$g(t) = g(0) e^{\lambda(e^u - 1)t} = e^{\lambda t(e^u - 1)}.$$

d) Deduce that $N_t \sim \mathcal{P}(\lambda t)$, and then (3).

Solution.

We proved that for any $t > 0$ and any $u \leq 0$,

$$\mathbb{E}\left[e^{uN_t}\right] = e^{\lambda t(e^u - 1)}.$$

We recognize the m.g.f. of a $\mathcal{P}(\lambda t)$ distributed r.v. (c.f. Proposition 1.2).

Finally, by the stationarity assumption, for all $s \geq 0$, $N_{s+t} - N_s$ and N_t have the same distribution, that is $\mathcal{P}(\lambda t)$.

□

II Arrival and interarrival times

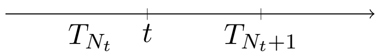
In all this section, N denotes a point process, and $(T_n)_{n\in\mathbb{N}^*}$ is the sequence of the corresponding arrival times (see Equations (2.1) and (2.2) for more details).

II.1 First arrival time after a given instant

Property 2.16.

Let N be a homogeneous Poisson process with rate $\lambda > 0$ and denote $(T_n)_{n\in\mathbb{N}^*}$ the corresponding arrival times.

- Then, the first arrival time satisfies $T_1 \sim \mathcal{E}(\lambda)$.
- Moreover, for all fixed $t > 0$, one has



$$T_{N_t} \leq t < T_{N_t+1}.$$

In particular, the first arrival time after a given instant t satisfies

$$T_{N_t+1} - t \sim \mathcal{E}(\lambda).$$

Proof. In exercise (c.f. Worksheet 2, Exercise 2). □

Example (The inspection paradox or *"le paradoxe de l'autobus"*).

You are waiting for a bus to arrive. It is written on your timetable that there is a bus every 10 minutes.

- If the arrival times of the bus where deterministic, and a bus arrived every 10 minutes, then the time you wait at the bus stop is uniformly distributed on $[0, 10]$ and, in expectation you wait for $10/2 = 5$ minutes.
- However, due to traffic, the arrival times of the buses are random. You assume they can be modeled by a Poisson process with rate $1/10$ (since in average, there is 1 bus in 10 min). Then, your waiting time equals $T_{N_t+1} - t$ which has and exponential distribution with parameter $1/10$. Hence, in average, you wait for 10 minutes (instead of the expected 5 mins).

II.2 Exponential interarrival times

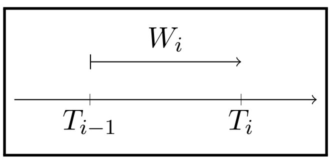
Definition 2.17.

The interarrival times (also called waiting times) of a point process are defined by

$$\begin{cases} W_1 = T_1, \\ W_i = T_i - T_{i-1}, \quad \forall i \geq 2. \end{cases}$$

The ith waiting time corresponds to the delay between the $(i-1)$th and the ith occurrences. In particular, for all n in $\mathbb{N}^*$,

$$T_n = \sum_{i=1}^{n} W_i.$$



Proposition 2.18.

Let N is a homogeneous Poisson process with rate λ.

1. *Then the corresponding interarrival times $(W_i)_{i\geq 1}$ are independent and identically distributed exponential random variables with parameter λ.*
2. *It follows that for all n in $\mathbb{N}^*$, the nth arrival time T_n has a gamma (or Erlang) distribution with parameters n and λ.*

Notice that, heuristically, the assumptions of stationarity and independent increments means that, at any point in time, the process *probabilistically* restarts itself, independently on the past. That is, the process from any time t is independent of all that has previously occurred, and also has the same distribution as the original process. In other words, the process has *no memory*, and hence, exponential interarrival times are to be expected (c.f. Section III of Chapter 1).

This result relies on the following Lemma.

Lemma 2.19.

Let N be a homogeneous Poisson process with rate λ, and denote $T_1 < T_2, \ldots < T_n < \ldots$ the corresponding arrival times. Then, for all n in $\mathbb{N}^*$, $(T_1, \ldots, T_n)$ has the following density w.r.t. the Lebesgue measure on $\mathbb{R}^n$:

$$(t_1, \ldots, t_n) \mapsto \lambda^n e^{-\lambda t_n} \mathbb{1}_{\{0<t_1<\ldots<t_n\}}.$$

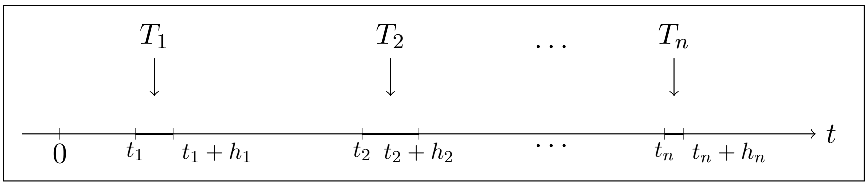
Idea of proof of Lemma 2.19.

- (Reminder) **For a real-valued continuous random variable:** Let X be a random variable with density f and c.d.f. F. Then, its density

$$f(x) = F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\mathbb{P}(X \in (t; t+h])}{h}.$$

- **In the multidimensional case:** Similarly, consider a strictly increasing sequence $0 < t_1 < \ldots < t_n$, and introduce for $h = (h_1, \ldots, h_n)$ small enough such that all $(t_i, t_i + h_i]$ are disjoint,

$$A_h = \{T_1 \in (t_1; t_1 + h_1], \ldots, T_n \in (t_n; t_n + h_n]\}.$$



Then,

$$f(t_1, \ldots, t_n) = \lim_{\forall i,\ h_i \to 0} \frac{\mathbb{P}(A_h)}{h_1 \ldots h_n}.$$

Yet, the event A_h is satisfied if and only if there are

- 0 points in $(0, t_1]$ and 1 point in $(t_1, t_1 + h_1]$,
- 0 points in $(t_1 + h_1, t_2]$ and 1 point in $(t_2, t_2 + h_2]$,
- ...

– 0 points in $(t_{n-1}+h_{n-1}, t_n]$ and 1 point in $(t_n, t_n+h_n]$.

Thus, since for $h_1, \ldots, h_n$ small enough, the intervals appearing in A_h are disjoints, by independence of the increments,

$$\begin{aligned}
\mathbb{P}(A_h) &= \mathbb{P}(N_{t_1}=0)\times\mathbb{P}(N_{t_1+h_1}-N_{t_1}=1)\times\mathbb{P}(N_{t_2}-N_{t_1+h_1}=0)\times\mathbb{P}(N_{t_2+h_2}-N_{t_2}=1)\times\\
&\quad\times\ldots\times\ \mathbb{P}\big(N_{t_n}-N_{t_{n-1}+h_{n-1}}=0\big)\times\mathbb{P}(N_{t_n+h_n}-N_{t_n}=1)\\
&= e^{-\lambda t_1}\times\left(\lambda h_1 e^{-\lambda h_1}\right)\times e^{-\lambda(t_2-[t_1+h_1])}\times\left(\lambda h_2 e^{-\lambda h_2}\right)\times\ldots\times e^{-\lambda(t_n-[t_{n-1}+h_{n-1}])}\times\left(\lambda h_n e^{-\lambda h_n}\right)\\
&= \lambda^n h_1\ldots h_n e^{-\lambda(t_n+h_n)}.
\end{aligned}$$

Hence,

$$f(t_1,\ldots,t_n)=\lim_{\forall i,\ h_i\to 0}\lambda^n e^{-\lambda(t_n+h_n)}=\lambda^n e^{-\lambda t_n}.$$

Note that, if we do not have $0<t_1<\ldots<t_n$, then $\mathbb{P}(A_h)=0$ since $0<T_1<\ldots<T_n$ a.s.

□

We can now prove Proposition 2.18.

Proof. In exercise.

1. Let N be a Poisson process with rate λ, and consider $(W_i)_{i\geq 1}$ the interarrival times. Let us prove that they are i.i.d. with distribution $\mathcal{E}(\lambda)$.

   (a) Fix $n\in\mathbb{N}^*$. Consider $g:\mathbb{R}^n\to\mathbb{R}^n$ a measurable bounded function. Prove that

$$\mathbb{E}[g(W_1,\ldots,W_n)]=\int_{\mathbb{R}^n} g(t_1,t_2-t_1,\ldots,t_n-t_{n-1})\lambda^n e^{-\lambda t_n}\mathbb{1}_{\{0<t_1<\ldots<t_n\}}dt_1\ldots dt_n.$$

Solution.

Consider the arrival times $T_i=\sum_{i=1}^n W_i$. Knowing the denisty of $(T_1,\ldots,T_n)$ given in Lemma 2.19 allows us to write

$$\begin{aligned}
\mathbb{E}[g(W_1,\ldots,W_n)] &= \mathbb{E}[g(T_1,T_2-T_1,\ldots,T_n-T_{n-1})]\\
&= \int_{\mathbb{R}^n} g(t_1,t_2-t_1,\ldots,t_n-t_{n-1})\lambda^n e^{-\lambda t_n}\mathbb{1}_{\{0<t_1<\ldots<t_n\}}dt_1\ldots dt_n
\end{aligned}$$

   (b) By a well-chosen change of variables, show that

$$\mathbb{E}[g(W_1,\ldots,W_n)]=\int_{\mathbb{R}^n} g(w_1,w_2,\ldots,w_n)\prod_{i=1}^n\left[\lambda e^{-\lambda w_i}\mathbb{1}_{\{w_i>0\}}\right]dw_1\ldots dw_n.$$

Solution.

Consider the following change of variable:

$$\begin{cases} w_1=t_1\\ w_2=t_2-t_1\\ \vdots\\ w_n=t_n-t_{n-1}\end{cases}\quad\Leftrightarrow\quad\begin{cases} t_1=w_1 & =\phi_1(w)\\ t_2=w_1+w_2 & =\phi_2(w)\\ \vdots & \vdots\\ t_n=\sum_{i=1}^n w_n & =\phi_n(w)\end{cases}$$

The function $\phi: w=(w_1,\ldots,w_n)\mapsto(w_1,w_1+w_2,\ldots,\sum_{i=1}^n w_i)$ is a one to one $\mathcal{C}^1$-

diffeomorphism with Jacobian matrix

$$J_\phi(w) = \begin{pmatrix} \frac{\partial \phi_1}{\partial w_1} & \cdots & \frac{\partial \phi_1}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial w_1} & \cdots & \frac{\partial \phi_n}{\partial w_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Hence, for all w in $\mathbb{R}^n$, $\det(J_\phi(w)) = 1 \neq 0$. Moreover,

$$\mathbb{1}_{\{0<t_1<\ldots<t_n\}} = \mathbb{1}_{\{\bigcap_{i=1}^n \{w_i>0\}\}} = \prod_{i=1^n} \mathbb{1}_{\{w_i>0\}}.$$

Hence, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} g(t_1, t_2 - t_1, \ldots, t_n - t_{n-1}) \lambda^n e^{-\lambda t_n} \mathbb{1}_{\{0<t_1<\ldots<t_n\}} dt_1 \ldots dt_n \\ &\quad = \int_{\mathbb{R}^n} g(w_1, w_2, \ldots, w_n) \lambda^n e^{-\lambda \sum_{i=1}^n w_i} \prod_{i=1^n} \mathbb{1}_{\{w_i>0\}} \left|\det(J_\phi(w))\right| dw_1 \ldots dw_n \\ &\quad = \int_{\mathbb{R}^n} g(w_1, w_2, \ldots, w_n) \underbrace{\prod_{i=1}^n \left[\lambda e^{-\lambda w_i} \mathbb{1}_{\{w_i>0\}}\right]}_{(*)} dw_1 \ldots dw_n. \end{aligned}$$

(c) Conclude.

Solution.

We recognize in $(*)$ the density of n i.i.d. r.v. with $\mathcal{E}(\lambda)$ distribution.

2. Immediate from Proposition 1.18, iii).

□

The reverse of Proposition 2.18 is true and thus provides a way of constructing a homogeneous Poisson process (see Section II.4). Before proving this, we need the notion of order statistics.

II.3 Conditional distribution of the arrival times

Definition 2.20 (Order statistic).

Let $Y_1, Y_2, \ldots, Y_n$ be n random variables. We say that $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the order statistics corresponding to $Y_1, \ldots, Y_n$ if $Y_{(k)}$ is kth smallest value among the Y_i's. In other words

$$Y(1) \leq Y_{(2)} \leq \ldots \leq Y_{(n)}.$$

As an example, if $n = 3$, $Y_1 = 4$, $Y_2 = 5$ and $Y_3 = 1$, then $Y_{(1)} = 1$, $Y_{(2)} = 4$ and $Y_{(3)} = 5$.

Note that even if the Y_i are independent and identically distributed, the order statistics are NOT independent, nor identically distributed.

Lemma 2.21.

If $Y_1, \dots, Y_n$ are i.i.d. continuous random variables with density f, then the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ is given by

$$(u_1, u_2, \dots, u_n) \mapsto n! \left(\prod_{i=1}^{n} f(u_i) \right) \mathbb{1}_{\{u_1 < u_2 < \dots < u_n\}}.$$

Proof. Denote $\mathfrak{S}_n$ the set of all permutations of $\{1, 2, \dots, n\}$. Let B be a Borel set in $\mathbb{R}^n$. Then

$$\mathbb{P}\big((Y_{(1)}, \dots, Y_{(n)}) \in B\big) = \sum_{\sigma \in \mathfrak{S}_n} \mathbb{P}\big(\{(Y_{\sigma(1)}, \dots, Y_{\sigma(n)}) \in B\} \cap \{Y_{\sigma(1)} < \dots < Y_{\sigma(n)}\}\big).$$

Yet, if $\sigma \in \mathfrak{S}_n$ is deterministic, then $Y_{\sigma(1)}, \dots, Y_{\sigma(n)}$ are alors i.i.d. with density f. Hence

$$\mathbb{P}\big((Y_{(1)}, \dots, Y_{(n)}) \in B\big) = \sum_{\sigma \in \mathfrak{S}_n} \int_{\mathbb{R}^n} \mathbb{1}_{\{(u_1, \dots, u_n) \in B\}} \mathbb{1}_{\{u_1 < \dots < u_n\}} f(u_1) \dots f(u_n) du_1 \dots du_n.$$

Finally, since the cardinality of $\mathfrak{S}_n$ equals $n!$,

$$\mathbb{P}\big((Y_{(1)}, \dots, Y_{(n)}) \in B\big) = \int_{\mathbb{R}^n} \mathbb{1}_{\{(u_1, \dots, u_n) \in B\}} \underbrace{\left[n! \left(\prod_{i=1}^{n} f(u_i) \right) \mathbb{1}_{\{u_1 < \dots < u_n\}} \right]}_{(*)} du_1 \dots du_n.$$

This being true for any B, we deduce that $(*)$ is the joint density of the order statistics $\big(Y_{(1)}, \dots, Y_{(n)}\big)$. □

In the particular case of uniform random variables, we obtain the following property.

Property 2.22.

i) If $U_1, \dots, U_n$ are i.i.d. r.v. with uniform distribution on $[0, t]$, then the joint density of the order statistics $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ is given by

$$(u_1, u_2, \dots, u_n) \mapsto \frac{n!}{t^n} \mathbb{1}_{\{0 < u_1 < u_2 < \dots < u_n < t\}}.$$

ii) We deduce the following integral

$$\int_{\mathbb{R}^n} \mathbb{1}_{\{0 < u_1 < u_2 < \dots < u_n < t\}} = \frac{t^n}{n!}.$$

Proof. Trivial from Lemma 2.21. □

The following result is very useful to prove the reverse of Proposition 2.18 and thus provides a way of sampling a homogeneous Poisson process. It can also be used to test for homogeneity of a Poisson process.

Proposition 2.23 (Conditional distribution).

Let N be a homogeneous Poisson process with rate $\lambda > 0$ and fix $t > 0$. Let $n \in \mathbb{N}^*$. Given that $N_t = n$, the n first arrival times $(T_1, \dots, T_n)$ have the same distribution as the order statistic corresponding to n independent random variables uniformly distributed on the interval $[0, t]$, that is

$$(T_1, T_2, \dots, T_n)|\{N_t = n\} \overset{(d)}{=} (U_{(1)}, U_{(2)}, \dots, U_{(n)}) \quad \text{where } U_1, \dots, U_n \text{ i.i.d } \sim \mathcal{U}([0, t]),$$

where $\overset{(d)}{=}$ means that both r.v. have the same distribution.

Proof. In exercise. Let $n \in \mathbb{N}^*$ and B be a Borel set.

a) Write the event $\{N_t = n\}$ in terms of the arrival times $(T_i)_i$.

Solution.

The event $\{N_t = n\}$ means that there are exactly n events by time t. This is equivalent to say that the nth event occurred by time t, and the $(n + 1)$th after time t. Hence

$$\{N_t = n\} = \{T_n \le t < T_{n+1}\}.$$

b) Using the definition of a Poisson process, and Lemma 2.19, deduce that

$$\begin{aligned}&\mathbb{P}((T_1, \dots, T_n) \in B|N_t = n)\\ &\quad= \frac{n!}{(\lambda t)^n e^{-\lambda t}} \int_{\mathbb{R}^{n+1}} \mathbb{1}_{\{(t_1,\dots,t_n)\in B\}} \lambda^{n+1} e^{-\lambda t^{n+1}} \mathbb{1}_{\{0<t_1<\dots<t_n\le t<t_{n+1}\}} dt_1 \dots dt_n dt_{n+1}.\end{aligned}$$

Solution.

By definition of the conditional probability,

$$\mathbb{P}((T_1, \dots, T_n) \in B|N_t = n) = \frac{\{\mathbb{P}((T_1, \dots, T_n) \in B) \cap \{N_t = n\}\}}{\mathbb{P}(N_t = n)}.$$

Yet, $N_t \sim \mathcal{P}(\lambda t)$, hence

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Moreover, $\{N_t = n\} = \{T_n \le t < T_{n+1}\}$. Thus,

$$\mathbb{P}((T_1, \dots, T_n) \in B|N_t = n) = \frac{n!}{(\lambda t)^n e^{-\lambda t}} \mathbb{P}(\{(T_1, \dots, T_n) \in B\} \cap \{T_n \le t < T_{n+1}\}).$$

Moreover, Lemma 2.19 provides the density of $(T_1, \dots, T_n, T_{n+1})$, directly leading to the desired formula.

c) Deduce that

$$\mathbb{P}((T_1, \dots, T_n) \in B|N_t = n) = \int_{\mathbb{R}^n} \mathbb{1}_{\{(t_1,\dots,t_n)\in B\}} \frac{n!}{t^n} \mathbb{1}_{\{0<t_1<\dots<t_n<t\}} dt_1 \dots dt_n.$$

Solution.

By simplification of the λ powers, and by the Fubini-Tonelli theorem, we can integrate first w.r.t t_{n+1}, and obtain

$$\begin{aligned}
&\mathbb{P}\big((T_1,\dots,T_n) \in B | N_t = n\big) \\
&\quad= \frac{n!}{t^n e^{-\lambda t}} \int_{\mathbb{R}^n} \mathbb{1}_{\{(t_1,\dots,t_n)\in B\}} \underbrace{\left[\int_t^{+\infty} \lambda e^{-\lambda t_{n+1}} dt_{n+1}\right]}_{e^{-\lambda t}} \mathbb{1}_{\{0<t_1<\dots<t_n\leq t\}} dt_1 \dots dt_n \\
&\quad= \int_{\mathbb{R}^n} \mathbb{1}_{\{(t_1,\dots,t_n)\in B\}} \frac{n!}{t^n} \mathbb{1}_{\{0<t_1<\dots<t_n<t\}} dt_1 \dots dt_n.
\end{aligned}$$

d) Conclude.

Solution.

We recognize the density of the order statistic of n i.i.d. uniformly distributed r.v. on $[0,t]$ given in Property 2.22.

□

II.4 Construction of a homogeneous Poisson process

Let us now prove the reverse of Proposition 2.18, which provides an alternative definition of a homogeneous Poisson process.

Theorem 2.24.

Let $(W_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\mathcal{E}(\lambda)$. Then the point process with arrival times defined for all $n \geq 1$ by $T_n = \sum_{i=1}^n W_i$, is a homogeneous Poisson process with rate λ.

This result relies on the following Lemmas.

Lemma 2.25.

Under the assumptions of Theorem 2.24, for all t in $\mathbb{R}_+$ and all n in $\mathbb{N}$,

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

In particular, for all $t > 0$, $N_t \sim \mathcal{P}(\lambda t)$.

Proof. Recall that $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$, where

$$\begin{cases} T_n = \sum_{i=1}^n W_i \sim \Gamma(n, \lambda), \\ T_{n+1} = T_n + W_{n+1} \quad \text{with} \quad W_{n+1} \perp\!\!\!\perp T_n, \text{ and } W_{n+1} \sim \mathcal{E}(\lambda). \end{cases}$$

1. Justify that

$$\mathbb{P}(N_t = n) = \int_{\mathbb{R}^2} \mathbb{1}_{\{s\leq t\}} \mathbb{1}_{\{s+w>t\}} \left[\frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s} \mathbb{1}_{\{s>0\}}\right] \left(\lambda e^{-\lambda w} \mathbb{1}_{\{w>0\}}\right) dsdw.$$

Solution.

From Definition 1.16, we have

$$\begin{aligned}\mathbb{P}(N_t = n) &= \mathbb{P}(\{T_n \le t\} \cap \{T_n + W_{n+1} > t\}) \\ &= \int_{\mathbb{R}^2} \mathbb{1}_{\{s \le t\}} \mathbb{1}_{\{s+w>t\}} \underbrace{\frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s} \mathbb{1}_{\{s>0\}}}_{\text{density of } T_n} \left(\lambda e^{-\lambda w} \mathbb{1}_{\{w>0\}} \right) ds dw.\end{aligned}$$

2. Compute $\mathbb{P}(N_t = n)$.

Solution.

Hence by the Fubini-Tonelli theorem,

$$\begin{aligned}\mathbb{P}(N_t = n) &= \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} \underbrace{\left[\int_{t-s}^{+\infty} \lambda e^{-\lambda w} dw \right]}_{e^{-\lambda(t-s)}} ds \\ &= \frac{\lambda^n}{(n-1)!} e^{-\lambda t} \underbrace{\int_0^t s^{n-1} ds}_{t^n/n} \\ &= \frac{(\lambda t)^n}{n!} e^{-\lambda t}.\end{aligned}$$

□

In order to prove the stationarity and the independence of the increments, we also need the joint distribution of the arrival times.

Lemma 2.26.

Under the assumptions of Theorem 2.24, the arrival times $(T_1, \ldots, T_n)$ have the following density w.r.t. the Lebesgue measure:

$$(t_1, \ldots, t_n) \mapsto \lambda^n e^{-\lambda t_n} \mathbb{1}_{\{0 < t_1 < \ldots < t_n\}}.$$

Note that we recover the same density as in Lemma 2.19 under the homogeneous Poisson process assumption (phew!).

Proof. "Same" change of variables as in the proof of Lemma 2.19. □

We may now prove Theorem 2.24.

Proof of Theorem 2.24. Notice that Lemmas 2.25 and 2.26 jointly imply that the conditional distribution of $(T_1, \ldots, T_n) | N_t = n$ is the same as the joint distribution of the order statistic $U_{(1)} < \ldots < U_{(n)}$ of i.i.d. r.v. $(U_1, \ldots, U_n)$ with distribution $\mathcal{U}([0, t])$ (exactly same proof as Proposition 2.23).

Let $k \in \mathbb{N}^*$ and $0 = t_0 < t_1 < \ldots < t_k = t$ be a subdivision of $[0, t]$.
Conditionally on $\{N_t = n\}$, since the $(T_1, \ldots, T_n) \overset{(d)}{=} (U_{(1)}, \ldots, U_{(n)})$,

$$N_{t_j} - N_{t_{j-1}} = \sum_{i=1}^n \mathbb{1}_{\{t_{j-1} < T_i \le t_j\}} \overset{(d)}{=} \sum_{i=1}^n \mathbb{1}_{\{t_{j-1} < U_{(i)} \le t_j\}} = \sum_{i=1}^n \mathbb{1}_{\{t_{j-1} < U_i \le t_j\}}.$$

Then $(N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_k} - N_{t_{k-1}})$ has a multinomial distribution with parameters $(n, (p_j)_{1\le j\le k})$ where

$$p_j = \mathbb{P}(t_{j-1} < U_1 \le t_j) = \frac{t_j - t_{j-1}}{t}, \qquad \text{since } U_1 \sim \mathcal{U}([0,t[).$$

Hence, for all integers $n_1, \ldots, n_k$ such that $\sum_{j=1}^k n_j = n$,

$$\begin{aligned}\mathbb{P}\big(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_{t_k} - N_{t_{k-1}} = n_k | N_t = n\big) &= \frac{n!}{n_1! \ldots n_k!} p_1^{n_1} \ldots p_k^{n_k} \\ &= \frac{n!}{t^n} \prod_{j=1}^k \frac{(t_j - t_{j-1})^{n_j}}{n_j!}.\end{aligned}$$

We thus obtain the unconditional probability

$$\begin{aligned}&\mathbb{P}\big(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_{t_k} - N_{t_{k-1}} = n_k\big) \\ &\quad = \mathbb{P}\big(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_{t_k} - N_{t_{k-1}} = n_k | N_t = n\big)\, \mathbb{P}(N_t = n) \\ &\quad = \frac{n!}{t^n} \prod_{j=1}^k \left[\frac{(t_j - t_{j-1})^{n_j}}{n_j!}\right] \times \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ &\quad = \prod_{j=1}^k \left(\frac{[\lambda(t_j - t_{j-1})]^{n_j}}{n_j!} e^{-\lambda(t_j - t_{j-1})}\right),\end{aligned}$$

since

$$\lambda^n = \lambda^{\sum_{j=1}^k n_j} = \prod_{j=1}^k \lambda^{n_j} \qquad \text{and} \qquad e^{-\lambda t} = e^{-\lambda \sum_{j=1}^k (t_j - t_{j-1})} = \prod_{j=1}^k e^{-\lambda(t_j - t_{j-1})}.$$

We recognize the joint distribution of k independent r.v. with respective distribution $\mathcal{P}(\lambda(t_j - t_{j-1}))$. This directly implies points (2) (independence of the increments) and (3) (the number of points in an interval $(t_{j-1}, t_j]$ has a Poisson distribution with parameter λ times the length $(t_j - t_{j-1})$ of the interval) in Definition 2.11. □

A possible generalization of Poisson processes is to consider a point process for which the times between successive events are independent and identically distributed (but not necessarily exponentially distributed). Such a point process is called *renewal process.*

III Divisibility of homogeneous Poisson processes

III.1 Two states divisibility

Proposition 2.27.

Consider a homogeneous Poisson process N with rate $\lambda > 0$ and assume that each time an event occurs it is classified as either a type I with probability p or a type II event with probability $(1-p)$ independently of all other events. Let N^{I} and N^{II} respectively denote the point processes corresponding to type I and type II events.

Then,

1. *For all $t > 0$, the total number of events occurring in $[0,t)$ equals $N_t = N_t^{\mathrm{I}} + N_t^{\mathrm{II}}$.*
2. *N^{I} and N^{II} are both Poisson processes having respective rates $p\lambda$ and $(1-p)\lambda$.*
3. *Furthermore, the two processes N^{I} and N^{II} are independent.*

This result lies on the following key point. The classification of each point being independent on when it occurs, we can say that the conditional distribution of N_t^{I} given that $N_t = n$ has a binomial distribution with parameters (n, p), that is

$$N_t^{\mathrm{I}} | \{N_t = n\} \sim \mathcal{B}(n, p). \tag{2.5}$$

Proof. Point 1 is trivial. Let us prove 2 and 3.

- The independence of the increments of N^{I} and N^{II} comes from the fact that N is a Poisson process. Hence the number of points in disjoint intervals are independent. Moreover, the type is decided independently on the point process.
- The stationarity of the increments of N^{I} and N^{II} also also comes from the fact that N is a Poisson process. Indeed, the number of points of N in two intervals with same length has the same distribution. Moreover, once again, the classification does not depend on the times the event occurs.
- **Poisson distribution and $\boldsymbol{N^{\mathrm{I}} \perp\!\!\!\perp N^{\mathrm{II}}}$:** Let $t > 0$ and $k, l \in \mathbb{N}$. Then,

$$\begin{aligned} \mathbb{P}\big(N_t^{\mathrm{I}} = k; N_t^{\mathrm{II}} = l\big) &= \mathbb{P}\big(N_t^{\mathrm{I}} = k; N_t = k+l\big) \\ &= \mathbb{P}\big(N_t^{\mathrm{I}} = k \big| N_t = k+l\big)\, \mathbb{P}(N_t = k+l)\,. \end{aligned}$$

Yet, from the key point stated above, the conditional distribution of N_t^{I} given $\{N_t = k+l\}$ is a binomial $\mathcal{B}(k+l, p)$. Moreover, $N_t \sim \mathcal{P}(\lambda t)$. Hence

$$\begin{aligned} \mathbb{P}\big(N_t^{\mathrm{I}} = k; N_t^{\mathrm{II}} = l\big) &= \frac{(k+l)!}{k!\; l!}\, p^k (1-p)^l\, \frac{(\lambda t)^{k+l}}{(k+l)!} e^{-\lambda t} \\ &= \underbrace{\left(\frac{(p\lambda t)^k}{k!} e^{-p\lambda t}\right)}_{\mathbb{P}(N_t^{\mathrm{I}}=k)} \underbrace{\left(\frac{((1-p)\lambda t)^l}{l!} e^{-(1-p)\lambda t}\right)}_{\mathbb{P}(N_t^{\mathrm{II}}=l)} \end{aligned}$$

We deduce that

$$\begin{cases} N_t^{\mathrm{I}} \sim \mathcal{P}(p\lambda t) \\ N_t^{\mathrm{II}} \sim \mathcal{P}((1-p)\lambda t) \\ N_t^{\mathrm{I}} \perp\!\!\!\perp N_t^{\mathrm{II}} \end{cases}$$

□

Example 2.28.

Immigrants to area A arrive at a Poisson rate of ten per week. Moreover, each immigrant is of English decent with probability 1/12. What is the probability that no people of English descent will emigrate to area A during the month of February?

Solution.

The number of English descents that emigrate to area A during February follows a Poisson distribution with probability 4 (weeks) $\times$ 10 (λ) $\times$ 1/12 (p) $=$ 10/3 so the answer is $e^{-10/3}$.

The reverse of Proposition 2.27 is true.

Proposition 2.29.

If N^{I} and N^{II} are two independent Poisson processes with respective rates λ_{I} and λ_{II}, then the counting process $N = (N_t)_{t \geq 0}$ defined for all t in $\mathbb{R}_+$ by

$$N_t = N_t^{\mathrm{I}} + N_t^{\mathrm{II}}$$

is also a Poisson process with rate $\lambda_{\mathrm{I}} + \lambda_{\mathrm{II}}$.

Proof. Facultative homework. □

III.2 Infinite divisibility

The following generalization of Proposition 2.27 can be easily proved by mathematical induction.

Proposition 2.30.

Let N be a homogeneous Poisson process with intensity λ. Assume each time an event occurs is associated to one of M types in $\{1, \ldots, M\}$ with probability $1/M$ independently of all other events. For all m in $\{1, \ldots, M\}$, denote N^m the point process associated to events of type m. Then $N^1, \ldots, N^m$ are m i.i.d. homogeneous Poisson processes with intensity λ/M.

Definition 2.31.

A random variable X is said to be infinitely divisible if for all M in N^*, there exists M i.i.d. random variables $Y_1, \ldots, Y_M$ such that X has the same distribution as $Y_1 + \ldots + Y_M$.

Property 2.32.

Random variables with distribution $\mathcal{P}(\lambda)$, $\mathcal{N}(\mu, \sigma^2)$, $\mathcal{E}(\lambda)$ or $\Gamma(n, \lambda)$ are infinitely divisible.

Property 2.33.

A homogeneous Poisson processes is infinitely divisible.