

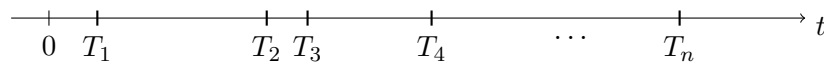
Statistics for homogeneous Poisson processes

In this part, we do some statistical inference for homogeneous Poisson processes. We observe a random point process $N = (N_t)_{t \in \mathbb{R}_+}$ with arrival times $(T_n)_{n \in \mathbb{N}^*}$. We assume that N is a homogeneous Poisson process with *unknown* intensity $\lambda > 0$. We aim at estimating λ . We also aim at constructing (asymptotic) confidence intervals for λ , and statistical tests.

I Introduction

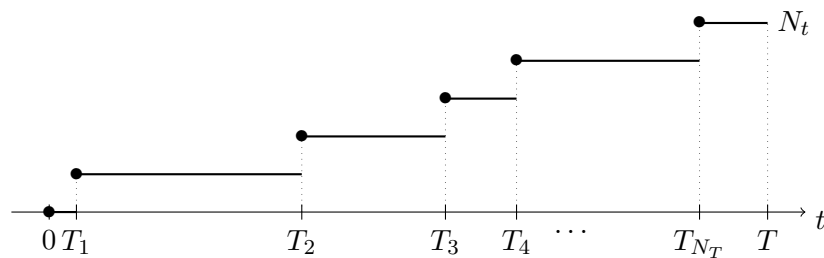
There are two types of observation for a point process:

- Either n is fixed, and we observe the process up to the n th event, that is we observe (T_1, \dots, T_n) .



In this case, the observation interval/window $[0, T_n]$ is random.

- Either the observation interval $[0, T]$ is fixed, and we observe the process on a fixed interval, that is we observe $(N_t)_{t \in [0, T]}$.



In this case, the number of events occurring in this interval N_T is random.

II Fixed number of points, random observation time

In this section n is fixed, and we observe a homogeneous Poisson process N with (unknown) intensity $\lambda > 0$ up to the n th arrival times. In particular, we observe the n first arrival times (T_1, \dots, T_n) or equivalently the n first interarrival times (W_1, \dots, W_n) . Then, according to Proposition 2.18, the $(W_i)_{i \geq 1}$ are i.i.d. with distribution $\mathcal{E}(\lambda)$ which makes the study easier. In particular, T_n is $\Gamma(n, \lambda)$ -distributed.

II.1 Maximum Likelihood Estimator (MLE)

Proposition 3.1.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$. Consider the n first arrival times (T_1, \dots, T_n) of N . Then likelihood of the trajectory of N is equal to

$$\mathcal{L}((T_1, \dots, T_n); \lambda) = \lambda^n e^{-\lambda T_n}.$$

Hence, the Maximum Likelihood Estimator (MLE) of λ is

$$\hat{\lambda}_n = \frac{n}{T_n}.$$

Proof. In exercise.

1. Recall the joint density of (T_1, \dots, T_n) , and deduce its likelihood.

2. Compute the MLE of λ .

□

Note that we could recover this result using that knowing the point process is also equivalent to knowing the interarrival times which are i.i.d. with $\mathcal{E}(\lambda)$ distribution. Indeed,

$$\mathcal{L}((W_1, \dots, W_n); \lambda) = \prod_{i=1}^n \left(\lambda e^{-\lambda W_i} \mathbf{1}_{\{W_i > 0\}} \right) = \lambda^n e^{-\lambda \sum_{i=1}^n W_i} = \lambda^n e^{-\lambda T_n}.$$

II.2 Non-asymptotic properties of the MLE

Lemma 3.2.

Let N be a homogeneous Poisson process with intensity $\lambda > 0$ and $n \geq 2$ a fixed integer. Then, the MLE of λ satisfies

$$\mathbb{E}[\hat{\lambda}_n] = \frac{n}{n-1}\lambda.$$

In particular, from Theorem 3.5, we can deduce that in the case "fixed n ", the Maximum Likelihood Estimator $\hat{\lambda}_n$ of λ is biased, and asymptotically unbiased.

However, we can consider

$$\tilde{\lambda} = \frac{n-1}{T_n},$$

and prove that $\tilde{\lambda}$ is an unbiased estimator of λ , but not efficient.

Proof. Worksheet 3, exercise 1. □

Moreover, to construct statistical tests with uniform asymptotic level, we need the following lemma.

Lemma 3.3.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$ observed up to the n th first arrival time T_n and consider the MLE $\hat{\lambda}_n = n/T_n$ of λ .

Then, for all (fixed) $t > 0$,

$$\lambda \mapsto \mathbb{P}_\lambda(\hat{\lambda}_n \geq t)$$

is a non-decreasing function.

Proof. In exercise. Consider $\lambda \leq \mu$ and prove that $\mathbb{P}_\lambda(\hat{\lambda}_n \geq t) \leq \mathbb{P}_\mu(\hat{\lambda}_n \geq t)$. *Hint:* Use Proposition 1.18.

□

II.3 Asymptotic distribution of the MLE

Before studying the asymptotic properties of the MLE, recall the univariate delta method.

Lemma 3.4 (Univariate delta method).

Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of real-valued random variables, Y be a real-valued random variable and θ be in \mathbb{R} such that

$$r_n(X_n - \theta) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} Y,$$

where (r_n) is a sequence of non-negative real numbers that tends toward infinity. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function at θ . Then,

$$f(X_n) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} f(\theta) \quad \text{and} \quad r_n(f(X_n) - f(\theta)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} f'(\theta)Y.$$

This allows us to deduce the asymptotic behavior of the MLE.

Theorem 3.5.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$ observed up to the n th first arrival time T_n and consider the MLE $\hat{\lambda}_n = n/T_n$ of λ .

1. **LLN-type result:** $\hat{\lambda}_n$ is a consistent estimator of λ , that is

$$\frac{n}{T_n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \lambda.$$

2. **CLT-type result:** Moreover, it is asymptotically gaussian, that is

$$\sqrt{n} \left(\frac{n}{T_n} - \lambda \right) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \lambda^2).$$

Comment 3.6. Note that here, the asymptotic variance equals λ^2 .

Proof. In exercise.

1. Apply the CLT to the interarrival times $(W_i)_{i \geq 1}$, and prove that

$$\sqrt{n} \left(\frac{1}{\hat{\lambda}_n} - \frac{1}{\lambda} \right) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0, \frac{1}{\lambda^2} \right). \quad (3.1)$$

2. Deduce Theorem 3.5 from the delta method.

□

II.4 Confidence intervals and statistical tests

Fix n in \mathbb{N}^* and let α in $(0, 1)$. We observe the n first arrival times of a homogeneous Poisson process with intensity $\lambda > 0$ and we aim at constructing confidence intervals for λ with confidence level $1 - \alpha$, or at testing, for instance,

$$\mathcal{H}_0 : \lambda \leq \lambda_0 \quad \text{against} \quad \mathcal{H}_1 : \lambda > \lambda_0, \quad \text{at level } \alpha.$$

A) Non-asymptotic statistical inference

Based on Proposition 1.18, one may construct a (non-asymptotic) confidence interval for λ with confidence level $1 - \alpha$, that is

$$\left[\frac{x_{2n, \alpha/2}}{2T_n}; \frac{x_{2n, 1-\alpha/2}}{2T_n} \right],$$

where $x_{d, \eta}$ denotes the η -quantile of a $\chi^2(d)$ distribution.

Proof. In exercise. □

Equivalently, one may prove that the test which rejects \mathcal{H}_0 if $2\lambda_0 T_n \leq x_{2n, \alpha}$ is of (non-asymptotic) level α , that is

$$\sup_{\lambda \leq \lambda_0} \{ \mathbb{P}_\lambda(2\lambda_0 T_n \leq x_{2n, \alpha}) \} \leq \alpha.$$

The test statistic of this test is

$$X_n := 2\lambda_0 T_n.$$

Moreover, if $\lambda = \lambda_0$, $X_n \sim \chi^2(2n)$. Hence, the p -value of this test equals

$$\mathbb{P}_{\lambda_0}(X_n \leq X_n^{obs}) = F_{2n}(X_n^{obs}),$$

where X_n^{obs} is the observed value of the test statistic (numerical application) and F_{2n} denotes the c.d.f. of a $\chi^2(2n)$ distribution.

Proof. Facultative homework. □

B) Asymptotic statistical inference

From Theorem 3.5, one can construct an asymptotic confidence interval that is

$$\left[\frac{n}{T_n \left(1 + \frac{z_{1-\alpha/2}}{\sqrt{n}}\right)}; \frac{n}{T_n \left(1 - \frac{z_{1-\alpha/2}}{\sqrt{n}}\right)} \right],$$

where $z_{1-\alpha/2}$ denotes the $(1 - \alpha/2)$ -quantile of a standard gaussian distribution $\mathcal{N}(0, 1)$.

Proof. In exercise □

Moreover, one can prove that the test which rejects \mathcal{H}_0 if $\hat{\lambda}_n \geq \lambda_0(1 + z_{1-\alpha}/\sqrt{n})$ is (uniformly) asymptotically of level α , that is

$$\limsup_{n \rightarrow +\infty} \left(\sup_{\lambda \leq \lambda_0} \left\{ \mathbb{P}_\lambda \left(\hat{\lambda}_n \geq \lambda_0 \left(1 + \frac{z_{1-\alpha}}{\sqrt{n}} \right) \right) \right\} \right) \leq \alpha.$$

This result lies on Lemma 3.3.

Note that, by isolating the quantile in the rejection rule, we recover the test statistic, that is

$$Z_n := \sqrt{n} \left(\frac{\hat{\lambda}_n}{\lambda_0} - 1 \right) = \sqrt{n} \left(\frac{n}{\lambda_0 T_n} - 1 \right).$$

Moreover, if $\lambda = \lambda_0$, $Z_n \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, 1)$. Hence the p -value of this test equals

$$\mathbb{P}_{\lambda_0}(Z_n \geq Z_n^{obs}) \approx 1 - \Phi(Z_n^{obs}),$$

where Z_n^{obs} denotes the observed value of the test statistic (numerical application), and Φ denotes the c.d.f. of a standard gaussian distribution $\mathcal{N}(0, 1)$.

Proof. Facultative homework. □

III Fixed window, random number of points

In this section T is fixed, and we observe a homogeneous Poisson process N with (unknown) intensity $\lambda > 0$ on the interval $[0, T]$. In particular, the number of "events" that have occurred by time T , that is N_T , is random.

III.1 Maximum Likelihood Estimator

Heuristic: Recall that, if X is a random variable with density f_θ and c.d.f. F_θ . Then, its likelihood in θ equals

$$\mathcal{L}(X, \theta) = f_\theta(X),$$

where,

$$f_\theta(x) = F'_\theta(x) = \lim_{h \rightarrow 0} \frac{F_\theta(x+h) - F_\theta(x)}{h} = \lim_{h \rightarrow 0} \frac{\mathbb{P}_\theta(X \in (t; t+h])}{h}.$$

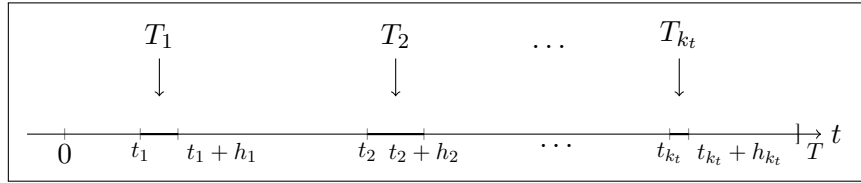
In the point process case: Recall that in the "fixed-T" case, $N_T = \sum_{i=1}^{+\infty} \mathbb{1}_{\{T_i \leq T\}}$ is random. Similarly to the proof of Lemma 2.19, consider a strictly increasing sequence $(t_i)_{i \in \mathbb{N}^*}$ in \mathbb{R}_+ , and denote

$$k_t = \sum_{i=1}^{+\infty} \mathbb{1}_{\{t_i \leq T\}} \quad \text{s.t.} \quad 0 =: t_0 < t_1 < \dots < t_{k_t} < T < t_{k_t+1} < \dots$$

Hence, k_t counts the number of t_i 's that belong to $[0, T]$.

Introduce for $h = (h_1, \dots, h_n)$ small enough such that all $(t_i, t_i + h_i]$ are disjoint (and $t_k + h_k \leq T$),

$$B_h = \{1 \text{ point in each } (t_i, t_i + h_i] \text{ and } 0 \text{ points elsewhere in } [0, T]\}.$$



The main difference with A_h in Lemma 2.19 is that now n is not fixed anymore, and we need to consider k_t which is the number of (t_i) which belong to the observation interval $[0, T]$.

Heuristically, as in the proof of Lemma 2.19,

$$f((t_i)_{i \in \mathbb{N}^*} \cap [0, T]) = \lim_{\forall i, h_i \rightarrow 0} \frac{\mathbb{P}(B_h)}{h_1 \dots h_{k_t}}.$$

Since the intervals appearing in B_h are disjoint, by independence of the increments,

$$\begin{aligned} \mathbb{P}(B_h) &= \left[\prod_{i=1}^{k_t} \mathbb{P}(1 \text{ point in } (t_i, t_i + h_i]) \right] \times \mathbb{P}(0 \text{ points elsewhere}) \\ &\stackrel{(*)}{=} \left[\prod_{i=1}^{k_t} \lambda h_i e^{-\lambda h_i} \right] \times e^{-\lambda(T - \sum_{i=1}^{k_t} h_i)} \\ &= \lambda^{k_t} h_1 \dots h_{k_t} e^{-\lambda T}. \end{aligned}$$

with $(*)$ is because "elsewhere" is of length $T - \sum_{i=1}^{k_t} h_i$.

Finally,

$$f((t_i)_{i \in \mathbb{N}^*} \cap [0, T]) = \lim_{\forall i, h_i \rightarrow 0} \lambda^{k_t} e^{-\lambda T} = \lambda^{k_t} e^{-\lambda T}.$$

Proposition 3.7.

Let N be a homogeneous Poisson process with (unknown) intensity λ . Consider the number N_T of "events" that have occurred in $[0, T]$, and denote (T_1, \dots, T_{N_T}) the corresponding arrival times. Then likelihood of the trajectory of N is equal to

$$\mathcal{L}((N_t)_{t \in [0, T]}; \lambda) = \lambda^{N_T} e^{-\lambda T}.$$

Hence, the Maximum Likelihood Estimator (MLE) of λ is

$$\hat{\lambda}_T = \frac{N_T}{T}.$$

Notice that the likelihood can be expressed as follows:

$$\mathcal{L}((N_t)_{t \in [0, T]}; \lambda) = \left[N_T! \left(\prod_{i=1}^{N_T} \frac{1}{T} \mathbb{1}_{\{T_i \in [0, T]\}} \right) \mathbb{1}_{\{T_1 < \dots < T_{N_T}\}} \right] \times \left[\frac{(\lambda T)^{N_T} e^{-\lambda T}}{N_T!} \right].$$

We recognize the product of

- the (conditional) likelihood of $(T_1, \dots, T_{N_T}) | N_T$, that is, according to Proposition 2.23, the likelihood of the order statistic corresponding to N_T independent random variables uniformly distributed on the interval $[0, T]$ given N_T ,
- and the likelihood of N_T which has a Poisson distribution $\mathcal{P}(\lambda T)$.

Proof. In exercise. Compute the MLE of λ .

The expression of the likelihood is admitted here (c.f. heuristic above).

□

III.2 Non-asymptotic properties of the MLE**Proposition 3.8.**

Let $(N_t)_{t \in [0, T]}$ be a Poisson process with rate $\lambda > 0$ observed on $[0, T]$ and consider the MLE $\hat{\lambda}_T = N_T/T$. Since $N_T \sim \mathcal{P}(\lambda T)$,

$$\mathbb{E}[\hat{\lambda}_T] = \lambda \quad \text{and} \quad \text{Var}(\hat{\lambda}_T) = \frac{\lambda}{T}.$$

We deduce that $\hat{\lambda}_T$ is an unbiased and efficient estimator of λ .

Proof. In exercise.

□

Moreover, to construct statistical tests with uniform asymptotic level, we need the following lemma.

Lemma 3.9.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$ observed on the interval $[0, T]$ and consider the MLE $\hat{\lambda}_T = N_T/T$ of λ .

Then, for all (fixed) $x \geq 0$,

$$\lambda \mapsto \mathbb{P}_\lambda(\hat{\lambda}_T \geq x)$$

is a non-decreasing function on \mathbb{R}_+^* .

Proof. In exercise. Fix $x \geq 0$ and denote

$$g : \lambda > 0 \mapsto \mathbb{P}_\lambda(\hat{\lambda}_T \geq x).$$

1. Assume $x = 0$. Compute $g(\lambda)$ for all $\lambda > 0$ and conclude in this case.

2. Now assume $x > 0$. Justify that for all $\lambda > 0$,

$$g(\lambda) = \sum_{k=\lceil xT \rceil}^{+\infty} \frac{(\lambda T)^k}{k!} e^{-\lambda T},$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

3. Deduce that

$$g'(\lambda) = \frac{(\lambda T)^{[xT]-1}}{([xT]-1)!} T e^{-\lambda T}.$$

Hint: Use the method of differences, that is telescoping series.

4. Conclude.

□

III.3 Asymptotic distribution of the MLE

Theorem 3.10.

Let $(N_t)_{t \in [0, T]}$ be a Poisson process with rate $\lambda > 0$ observed on $[0, T]$ and consider the MLE $\hat{\lambda}_T = N_T/T$.

1. LLN-type result: $\hat{\lambda}_T$ is a consistent estimator of λ , that is

$$\frac{N_T}{T} \xrightarrow[T \rightarrow +\infty]{\mathbb{P}} \lambda.$$

2. CLT-type result: Moreover, it is asymptotically gaussian, that is

$$\sqrt{T} \left(\frac{N_T}{T} - \lambda \right) \xrightarrow[T \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \lambda).$$

Note that it is also possible to prove that $N_T \xrightarrow[T \rightarrow +\infty]{a.s.} +\infty$ and deduce the "strong" version of the LLN-type result:

$$\frac{N_T}{T} \xrightarrow[T \rightarrow +\infty]{a.s.} \lambda.$$

This result is based on the following inequality: $\frac{T_{N_T}}{N_T} \leq \frac{T}{N_T} < \frac{T_{N_T+1}}{N_T}$.

Proof. In exercise.

1. Note that we cannot apply the classical Law of Large Numbers since the index in the limit is not a countable sequence. Apply Chebychev's inequality to prove the consistency of $\hat{\lambda}_T$.

2. Prove that the characteristic function of $Z_T := \sqrt{T}(N_t/T - \lambda)$ equals for all $u \in \mathbb{R}$,

$$\mathbb{E}[e^{iuZ_T}] = \exp\left(\lambda T \left[e^{\frac{iu}{\sqrt{T}}} - 1 - \frac{iu}{\sqrt{T}} \right]\right).$$

Hint: Use Proposition 1.2.

3. Prove that

$$\lim_{T \rightarrow +\infty} \mathbb{E}[e^{iuZ_T}] = e^{-\frac{\lambda u^2}{2}}.$$

Hint: Use Taylor's expansion.

4. Deduce the asymptotic distribution of Z_T . *Hint:* Recall that the characteristic function of a gaussian r.v. Z with distribution $\mathcal{N}(m, \sigma^2)$ equals for all $u \in \mathbb{R}$, $\mathbb{E}[iuZ] = \exp\left(imu - \frac{u^2\sigma^2}{2}\right)$.

□

III.4 Confidence intervals and statistical tests

Fix $T > 0$ and let α in $(0, 1)$. We observe a homogeneous Poisson process with unknown intensity $\lambda > 0$ on $[0, T]$ and we aim at constructing confidence intervals for λ with confidence level $1 - \alpha$, or at testing, for instance,

$$\mathcal{H}_0 : \lambda \leq \lambda_0 \quad \text{against} \quad \mathcal{H}_1 : \lambda > \lambda_0, \quad \text{at level } \alpha.$$

From Theorem 3.10 and Slutsky's Lemma, for all α in $(0, 1)$, we construct an asymptotic confidence interval for λ with confidence level $1 - \alpha$, that is

$$\left[\frac{N_T}{T} - \frac{\sqrt{N_T}}{T} z_{1-\alpha/2}; \frac{N_T}{T} + \frac{\sqrt{N_T}}{T} z_{1-\alpha/2} \right],$$

where $z_{1-\alpha/2}$ denotes the $(1 - \alpha/2)$ -quantile of a standard gaussian distribution $\mathcal{N}(0, 1)$.

Proof. In exercise. □

Moreover, one can construct different test.

$$\textbf{Test 1:} \text{ Reject } \mathcal{H}_0 \text{ if } \hat{\lambda}_T \geq \lambda_0 + \sqrt{\frac{\lambda_0}{T}} z_{1-\alpha}, \quad \text{with} \quad \begin{cases} \text{test statistic: } Z_T^{(1)} = \sqrt{T} \frac{\hat{\lambda}_T - \lambda_0}{\sqrt{\lambda_0}}, \\ p\text{-value: } 1 - \Phi(Z_T^{(1),obs}), \end{cases}$$

$$\textbf{Test 2:} \text{ Reject } \mathcal{H}_0 \text{ if } \hat{\lambda}_T \geq \lambda_0 + \sqrt{\frac{\hat{\lambda}_T}{T}} z_{1-\alpha}, \quad \text{with} \quad \begin{cases} \text{test statistic: } Z_T^{(2)} = \sqrt{T} \frac{\hat{\lambda}_T - \lambda_0}{\sqrt{\hat{\lambda}_T}}, \\ p\text{-value: } 1 - \Phi(Z_T^{(2),obs}), \end{cases}$$

where $z_{1-\alpha}$ and Φ respectively denote the $(1 - \alpha)$ -quantile and the c.d.f. of a standard gaussian distribution $\mathcal{N}(0, 1)$.

One can prove that both Test 1 and Test 2 are (uniformly) asymptotically of level α , that is

$$\limsup_{n \rightarrow +\infty} \left(\sup_{\lambda \leq \lambda_0} \{ \mathbb{P}_\lambda(\text{reject } \mathcal{H}_0) \} \right) \leq \alpha.$$

This results relies on Lemma 3.9.

Proof. Facultative homework. □

Take home message

Estimator	Unbiased	Efficient	Asymptotically unbiased	Consistent	Asymptotically gaussian
$\hat{\lambda}_n = \frac{n}{T_n}$	\times	.	\checkmark	\checkmark	$\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \lambda^2)$
$\tilde{\lambda}_n = \frac{n-1}{T_n}$	\checkmark	\times	\checkmark	\checkmark	(?)
$\hat{\lambda}_T = \frac{N_T}{T}$	\checkmark	\checkmark	\checkmark	\checkmark	$\sqrt{T}(\hat{\lambda}_T - \lambda) \xrightarrow[T \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \lambda)$

