

Inhomogeneous Poisson processes

In this chapter, we consider inhomogeneous Poisson processes. They generalize homogeneous Poisson processes by allowing the rate of the process to be a varying function of time. In particular, by doing so, we drop the stationary increments property.

I Definitions and basic properties

Definition 4.1.

Let $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a real valued piecewise continuous (or at least locally integrable) function defined on \mathbb{R}_+ . A point process $(N_t)_{t \in \mathbb{R}_+}$ is said to be an inhomogeneous Poisson process with intensity function λ if

- (1) $N_0 = 0$.
- (2) The process has independent increments.
- (3) For all $t > 0$, the number of points in $[0, t]$ has a Poisson distribution with parameter $\int_0^t \lambda(u) du$, that is

$$N_t \sim \mathcal{P} \left(\int_0^t \lambda(u) du \right).$$

The function $\Lambda : t \mapsto \int_0^t \lambda(u) du$ is called *cumulative intensity* of the Poisson process.

Proposition 4.2.

Let N be an inhomogeneous Poisson process with intensity function λ . Denote for all Borel subset A of \mathbb{R}_+ , $N(A)$ the number of points in A . Note that if $(T_i)_{i \in \mathbb{N}^*}$ denote the arrival times, then,

$$N(A) = \sum_{n \in \mathbb{N}^*} \mathbf{1}_{\{T_n \in A\}}$$

1. For all bounded Borel set A in \mathbb{R}_+ , $N(A)$ is a.s. finite.
2. For all $s \geq 0$ and all $t > 0$, the random variable $N([s, s+t]) = N_{s+t} - N_s$ has a Poisson distribution with parameter $\mu([s, s+t]) = \int_s^{s+t} \lambda(u) du$, that is

$$N_{s+t} - N_s \sim \mathcal{P} \left(\int_s^{s+t} \lambda(u) du \right).$$

Proof. 1. Let A be a bounded Borel set in \mathbb{R}_+ . Hence, there exists $t_A \in \mathbb{R}_+^*$ s.t. $A \subset [0, t_A]$. Therefore,

$$N(A) = \sum_{i \in \mathbb{N}^*} \mathbb{1}_{\{T_n \in A\}} \leq \sum_{i \in \mathbb{N}^*} \mathbb{1}_{\{T_n \leq t_A\}} = N_{t_A}.$$

Moreover, $N_{t_A} \sim \mathcal{P}(\int_0^{t_A} \lambda(u) du)$. Hence $\mathbb{E}[N_{t_A}] = \int_0^{t_A} \lambda(u) du < +\infty$. We deduce that, N_{t_A} is finite a.s., and thus, so is $N(A)$.

2. In exercise:

(a) Recall the moment generating function (or m.g.f.), or Laplace transform, of a Poisson r.v., and for all $t > 0$, express the m.g.f. of N_t w.r.t. the cumulative intensity Λ of the process.

(b) Let $s \geq 0$ and $t > 0$. Compute the m.g.f. of $N_{s+t} - N_s$, and conclude. *Hint:* Use $N_{s+t} = N_{s+t} - N_s + N_s$.

□

Let us notice several points.

- a) If for all $t \geq 0$, $\lambda(t) = \lambda$, that is λ is a constant function, $\int_s^{s+t} \lambda du = \lambda t$, and one recovers the definition of a homogeneous Poisson process with rate λ .
- b) Notice that Definition 4.1 characterizes the distribution of the process N . Indeed, for any n and for any $0 \leq t_1 < t_2 < \dots < t_n$, Definition 4.1 and Proposition 4.2 both provide the distribution of $(N_{t_1}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}})$. Hence, the distribution of the process $(N_t)_{t \in \mathbb{R}_+}$ is well known.
- c) The terminology of point process theory has been criticized for being too varied. In addition to the word point often being omitted, the homogeneous Poisson (point) process is also called a

stationary Poisson (point) process, as well as uniform Poisson (point) process. The inhomogeneous Poisson (point) process, as well as being called nonhomogeneous, is also referred to as the non-stationary Poisson process.

- d) The positive measure μ defined on any Borel set A by $\mu(A) = \int_A \lambda(u)du$ (or equivalently $d\mu(u) = \lambda(u)du$) is called the *intensity measure* of the Poisson process N . This definition of an inhomogeneous Poisson process can be generalized to any non-negative intensity measure on \mathbb{R}_+ that is locally bounded (i.e. for all bounded Borel subset A in \mathbb{R}_+ , $\mu(A) < +\infty$), not necessarily absolutely continuous with respect to the Lebesgue measure by replacing assumption (3) by for all Borel set A in \mathbb{R}_+ , $N(A) \sim \mathcal{P}(\int_A d\mu)$.

However, in this lecture, we restrict attention to the case $d\mu(u) = \lambda(u)du$, that is Poisson processes with intensity functions.

Property 4.3.

Let N be a homogeneous Poisson process with intensity function λ on \mathbb{R}_+ .

1. The mean function of N equals

$$\forall t \geq 0, \quad m(t) = \mathbb{E}[N_t] = \int_0^t \lambda(u)du = \Lambda(t).$$

In particular, the rate (see Definition 2.7) of the Poisson process N equals:

$$\forall t \geq 0, \quad w(t) = m'(t) = \lambda(t).$$

2. Moreover, for all $t \geq 0$, as $h \rightarrow 0$,

- (a) $\mathbb{P}(N_{t+h} - N_t = 0) = 1 - \lambda(t)h + o(h)$.
- (b) $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda(t)h + o(h)$.
- (c) $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$.

Proof. In exercise.

1. Immediate.
2. To do. *Hint:* Since $\Lambda'(t) = \lambda(t)$, one has $\Lambda(t+h) - \Lambda(t) = \lambda(t)h + o(h)$.

□

As in the homogeneous case, one can provide an equivalent definition for inhomogeneous Poisson processes.

Definition 4.4.

The point process N is said to be an inhomogeneous Poisson process with intensity function λ defined on if

- (1) $N_0 = 0$.
- (2) The process has independent increments.
- (3) For all positive t , one has as $h \rightarrow 0$,
 - (a) $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda(t)h + o(h)$.
 - (b) $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$.

The proof is the same as in the homogeneous case, except that we obtain a linear differential equation with non-constant coefficients.

Notice that, a Poisson process N with intensity function λ is regular since, for all $t > 0$,

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(N_{t+h} - N_t \geq 2)}{h} = 0.$$

In particular, it is a simple process, that is no more than 1 event can occur at the same time, and each jump of the counting process equals 1 (almost surely). Notice that it is not the case anymore when we generalize to inhomogeneous Poisson processes with non-diffuse intensity measure (i.e. that contains atoms).

II Construction of an inhomogeneous Poisson process

II.1 Time change

A first way of constructing an inhomogeneous Poisson process with intensity function λ on \mathbb{R}_+ (and thus proving that such processes exist) is by changing the time scale of a homogeneous Poisson process. Before, let us recall the definition and some properties of the generalized inverse function.

A) Generalized inverse function

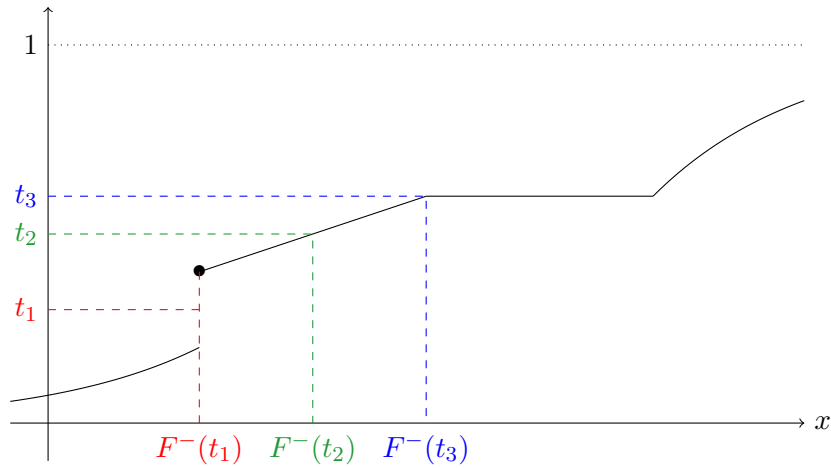
Definition 4.5.

Let $F : \mathbb{R} \rightarrow \mathbb{R}_+$ be a non-decreasing càdlàg non-negative function defined on \mathbb{R} . Its generalized inverse function is defined by

$$F^- : \left(\begin{array}{ll} \mathbb{R}_+ & \longrightarrow \mathbb{R} \cup \{\pm\infty\} \\ t & \longmapsto \inf \{x \in \mathbb{R} ; F(x) \geq t\} \end{array} \right),$$

with the conventions $\inf(\emptyset) = +\infty$ and $\inf(\mathbb{R}) = -\infty$.

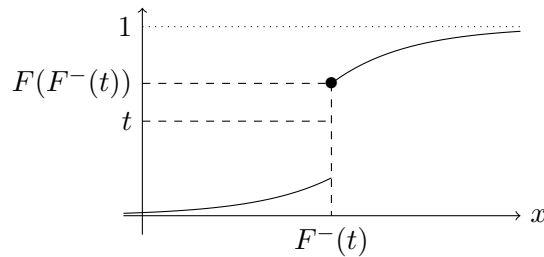
Note that if F is a c.d.f., then F^- is the corresponding quantile function.



Property 4.6.

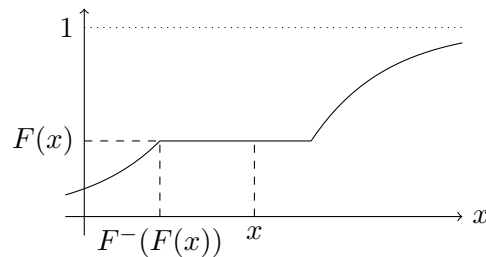
The generalized inverse function satisfies the following properties.

- i. F^- is a non-decreasing function.
- ii. $F(x) \geq t \iff x \geq F^-(t)$.
- iii. F^- is left-continuous, that is for all t_0 in \mathbb{R} , $\lim_{\substack{t \rightarrow t_0 \\ t < t_0}} F^-(t) = F(t_0)$.
- iv. For all $t \in \mathbb{R}_+$, $F(F^-(t)) \geq t$.



Moreover, if F is continuous, for all $t \in \mathbb{R}_+$, $F(F^-(t)) = t$.

- v. For all $x \in \mathbb{R}$, $F^-(F(x)) \leq x$.



Moreover, if F is strictly increasing, for all $x \in \mathbb{R}$, $F^-(F(x)) = x$.

- vi. In particular, if F is strictly increasing and continuous, then F is bijective with inverse function $F^{-1} = F^-$.

Proof. Denote for all $t \in \mathbb{R}^+$,

$$A_t = \{x \in \mathbb{R} ; F(x) \geq t\},$$

such that $F^-(t) = \inf(A_t)$.

- i. If $s \leq t$, then, for all $x \in \mathbb{R}$, $F(x) \geq t \Rightarrow F(x) \geq s$. Hence, $A_t \subset A_s$ and thus,

$$F^-(t) = \inf(A_t) \geq \inf(A_s) = F^-(s).$$

- ii. \Rightarrow If $F(x) \geq t$, then $x \in A_t$ and thus $F^-(t) = \inf(A_t) \leq x$.

\Leftarrow Assume $x \geq F^-(t)$. Let $\varepsilon > 0$. Then $x + \varepsilon > x \geq F^-(t)$, so $x + \varepsilon > \inf(A_t)$.

In particular, there exists $x_\varepsilon \in A_t$ such that $x_\varepsilon \leq x + \varepsilon$. Since F is non-decreasing, then $F(x + \varepsilon) \geq F(x_\varepsilon) \geq t$, as $x_\varepsilon \in A_t$.

We proved that, for all $\varepsilon > 0$, $F(x + \varepsilon) \geq t$. Hence, since F is right-continuous, as $\varepsilon \rightarrow 0^+$,

$$F(x) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} F(x + \varepsilon) \geq t.$$

- iii. Admitted here. Note that F^- is not right-continuous as soon as F is not strictly increasing.

- iv. By applying ii. to $x = F^-(t)$ directly leads to $F(F^-(t)) = F(x) \geq t$.

Assume now that F is continuous. Consider $(x_n)_n$ a sequence satisfying

$$\forall n, \quad x_n < F^-(t) \quad \text{and} \quad \lim_{n \rightarrow +\infty} x_n = F^-(t).$$

Then by contraposition of ii., for all n , $F(x_n) < t$, and by taking $n \rightarrow +\infty$, as F is continuous (and in particular left-continuous),

$$F(F^-(t)) = \lim_{n \rightarrow +\infty} F(x_n) \leq t.$$

Finally, in this case, both $F(F^-(t)) \leq t$ and $F(F^-(t)) \geq t$ imply that $F(F^-(t)) = t$.

- v. By applying ii. to $t = F(x)$ leads to $F^-(F(x)) = F^-(t) \leq x$.

Assume now that F is strictly increasing. Assume (absurd) that $F^-(F(x)) < x$. Then

$$F(F^-(F(x))) < F(x),$$

which is impossible by the first point applied to $t = F(x)$.

- vi. Immediate from the previous point.

□

B) Back to inhomogeneous Poisson processes

Proposition 4.7.

Let $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a locally integrable function, and denote

$$\Lambda : x \in \mathbb{R}_+ \mapsto \int_0^x \lambda(u) du.$$

Note that Λ is a non-decreasing continuous function.

1. Let \tilde{N} be a homogeneous Poisson process with (constant) rate 1, and denote

$$0 < S_1 < S_2 < \dots < S_n < \dots$$

its arrival times. Define respectively for all $n \in \mathbb{N}^*$, and for all $t \in \mathbb{R}_+$

$$T_n = \Lambda^-(S_n) \quad \text{and} \quad N_t = \sum_{n \in \mathbb{N}^*} \mathbf{1}_{\{T_n \leq t\}},$$

where Λ^- denote the generalized inverse function of Λ . Then $N = (N_t)_{t \in \mathbb{R}_+}$ is an inhomogeneous Poisson process with intensity function λ .

2. Conversely, assume that Λ is strictly increasing (and thus a bijective function, since it is always continuous). Let N be a inhomogeneous Poisson process with intensity function λ on \mathbb{R}_+ . Denote

$$0 < T_1 < T_2 < \dots < T_n < \dots,$$

its arrival times. Define respectively for all n in \mathbb{N}^* , and for all t in \mathbb{R}_+ ,

$$S_n = \Lambda(T_n) \quad \text{and} \quad \tilde{N}_t = \sum_{n \in \mathbb{N}^*} \mathbf{1}_{\{S_n \leq t\}}.$$

Then the point process $\tilde{N} = (\tilde{N}_t)_{t \in \mathbb{R}_+}$ is a homogeneous Poisson process with (constant) rate 1.

Proof. In exercise.

1. (a) Let $t > 0$. Express N_t in terms of $(S_n)_{n \in \mathbb{N}^*}$, and deduce, by Property 4.6, that $N_t = \tilde{N}_{\Lambda(t)}$.

(b) Conclude.

2. Assume that Λ is strictly increasing (and thus bijective), and denote Λ^{-1} its inverse function.

(a) What is the monotony of Λ^{-1} ? Deduce that for all u, t ,

$$\Lambda(u) \leq t \iff u \leq \Lambda^{-1}(t).$$

(b) Deduce that for all $t > 0$, $\tilde{N}_t = N_{\Lambda^{-1}(t)}$.

(c) Conclude.

□

II.2 Acceptance/Rejection (or Thinning)

The following result leads to one of the most popular method for generating an inhomogeneous Poisson process, sometimes called the *thinning method*. It is the "point process" analogue of the acceptance/rejection method classically used to simulate r.v.

It generalizes Proposition 2.27 which claims that if you randomly classify each point of a homogeneous Poisson process with rate λ into two categories (with probability p and $1 - p$), the two corresponding point processes are also homogeneous Poisson processes with corresponding rates $p\lambda$ and $(1 - p)\lambda$. In the inhomogeneous case, we allow p to be a varying function of time.

Theorem 4.8.

Let $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ be a locally integrable non-negative function. Assume that λ is upper-bounded by $M \in \mathbb{R}_+^*$ on a subset $[0, T)$ of \mathbb{R}_+ ($T \in \mathbb{R}_+^* \cup \{+\infty\}$), that is

$$\forall u \in [0, T), \quad \lambda(u) \leq M.$$

Let N be a homogeneous Poisson process with rate M and classify each point u of N as type I with probability $p(u) := \lambda(u)/M$ and type II otherwise. Then, the resulting point processes N^I and N^{II} of type I and type II events satisfy

- a) N^I is a inhomogeneous Poisson process with intensity function λ .
 N^{II} is a inhomogeneous Poisson process with intensity function $M - \lambda$.
- b) $N^I \perp\!\!\!\perp N^{II}$.

Idea of proof. a) The independence of the increments comes from the independence of the increments of the homogeneous Poisson process N , and the independence between the tags.

b) Let us now prove the Poisson distribution, and that $N^I \perp\!\!\!\perp N^{II}$.

- **Modeling:** We know (c.f. Proposition 2.23) that conditionally on $\{N_t = n\}$, T_1, \dots, T_n behaves as the order statistic $U_{(1)} \leq \dots \leq U_{(n)}$ associated to n i.i.d. r.v. U_1, \dots, U_n with distribution $\mathcal{U}([0, T])$. The classifying process can be seen as follows: Let Y_1, \dots, Y_n be n i.i.d. r.v. with distribution $\mathcal{U}([0, M])$, independent on (U_1, \dots, U_n) .

$$\text{Classify } U_i \text{ as } \begin{cases} \text{type I if } Y_i \leq \lambda(U_i), \\ \text{type II otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \mathbb{P}(U_i \text{ of type I} | U_i = u) &= \mathbb{P}(Y_i \leq \lambda(U_i) | U_i = u) \\ &= \mathbb{P}(Y_i \leq \lambda(u) | U_i = u) \\ &= \mathbb{P}(Y_i \leq \lambda(u)), && \text{as } Y_i \perp\!\!\!\perp U_i \\ &= \frac{\lambda(u)}{M} = p(u) && \text{as } Y_i \sim \mathcal{U}([0, M]). \end{aligned}$$

- **Towards a binomial distribution:** Conditionally on $\{N_t = n\}$, we thus obtain

$$N_t^I = \sum_{i=1}^n \mathbf{1}_{\{Y_i \leq \lambda(U_i)\}},$$

where the $\mathbb{1}_{\{Y_i \leq \lambda(U_i)\}}$ are independent (by independence of the couples $(U_i, Y_i)_i$) Bernoulli r.v. with parameter

$$\tilde{p} := \mathbb{P}(Y_i \leq \lambda(U_i)) = \int_0^{+\infty} \underbrace{\mathbb{P}(Y_i \leq \lambda(U_i) | U_i = u)}_{p(u)} \underbrace{f_{U_i}(u)}_{\frac{1}{t} \mathbb{1}_{\{u \in [0, t]\}}} du = \frac{1}{t} \int_0^t p(u) du.$$

We deduce that the conditional distribution of N_t^I given $\{N_t = n\}$, is a binomial distribution with parameters n and \tilde{p} , that is

$$N_t^I \sim \mathcal{B}(n, \tilde{p}).$$

- **Unconditioning:** Let $k, l \in \mathbb{N}$. Then

$$\begin{aligned} \mathbb{P}(N_t^I = k, N_t^{II} = l) &= \mathbb{P}(N_t^I = k, N_t = k + l) \\ &= \mathbb{P}(N_t^I = k | N_t = k + l) \mathbb{P}(N_t = k + l) \\ &= \binom{k+l}{k} \tilde{p}^k (1 - \tilde{p})^l \times \frac{(Mt)^{k+l}}{(k+l)!} e^{-Mt(\tilde{p}+1-\tilde{p})} \\ &= \left[\frac{(Mt\tilde{p})^k}{k!} e^{-Mt\tilde{p}} \right] \times \left[\frac{(Mt[1-\tilde{p}])^l}{l!} e^{-Mt(1-\tilde{p})} \right]. \end{aligned}$$

We deduce that

$$\mathbb{P}(N_t^I = k) = \sum_{l=0}^{+\infty} \mathbb{P}(N_t^I = k, N_t^{II} = l) = \frac{(Mt\tilde{p})^k}{k!} e^{-Mt\tilde{p}},$$

thus

$$N_t^I \sim \mathcal{P}(Mt\tilde{p}), \quad \text{with} \quad Mt\tilde{p} = Mt \frac{1}{t} \int_0^t \frac{\lambda(u)}{M} du = \int_0^t \lambda(u) du.$$

In the same way, we prove that $N_t^{II} \sim \mathcal{P}(Mt[1-\tilde{p}])$, with

$$Mt[1-\tilde{p}] = Mt - \int_0^t \lambda(u) du = \int_0^t [M - \lambda(u)] du.$$

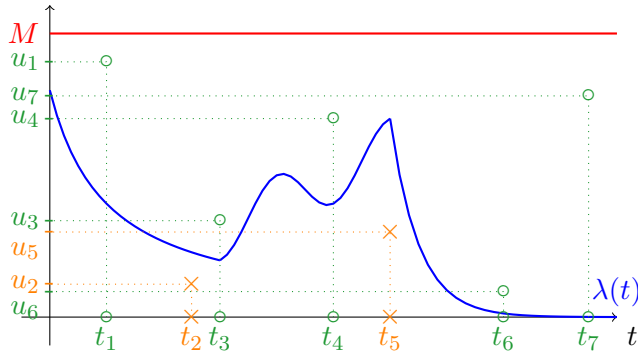
Finally, we also deduce that $N_t^I \perp\!\!\!\perp N_t^{II}$ because of that product form above. □

Algorithm 1 *Thinning algorithm*

Input: intensity function λ ; simulation interval $[0, T]$; upper-bound $M > 0$.

1. Generate a homogeneous Poisson process with constant rate M on $[0, T]$;
Denote t_1, \dots, t_n the observed arrival times on $[0, T]$.
2. Generate independently n i.i.d r.v. with distribution $\mathcal{U}[0, M]$;
Denote u_1, \dots, u_n the observed values.
3. for i in 1 to n , keep t_i if $u_i \leq \lambda(t_i)$.
4. Order the kept points.

Output: the set of ordered kept points are the arrival times of an inhomogeneous Poisson process with intensity function λ on $[0, T]$.



Observed arrival times:

- N : t_1, \dots, t_7 ;
- N^I : t_2, t_5 ;
- N^{II} : t_1, t_3, t_4, t_6, t_7 .

III Arrival times

III.1 First arrival time after a given instant

Property 4.9.

Let N be a Poisson process with intensity function $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

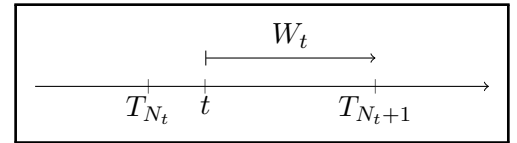
- i. Then the survival function of the first arrival time T_1 equals for all $s > 0$,

$$\mathbb{P}(T_1 > s) = \mathbb{P}(N_s = 0) = e^{-\int_0^s \lambda(u) du} = e^{-\Lambda(s)}.$$

It is not exponentially distributed anymore (except if $\Lambda(s) = \lambda s$, i.e. if the Poisson process is homogeneous). Note that λ is the hazard rate of T_1 .

- ii. Denote W_t the waiting time at time t , that is

$$W_t = T_{N_t+1} - t.$$



Then, for all $s > 0$,

$$\mathbb{P}(W_t > s) = \mathbb{P}(N_{t+s} - N_t = 0) = e^{-\int_t^{t+s} \lambda(u) du} = e^{-\int_0^s \lambda(u+t) du}.$$

In particular, from Proposition 1.14, the hazard rate of W_t is

$$h_t : u \mapsto \lambda(u + t).$$

III.2 (Conditional) distribution of the arrival times

Proposition 4.10.

Let N be an inhomogeneous Poisson process with intensity function $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Fix $n \in \mathbb{N}^*$ and denote (T_1, \dots, T_n) the n first arrival times. Then (T_1, \dots, T_n) has the following joint density w.r.t. the Lebesgue measure on \mathbb{R}^n :

$$(t_1, \dots, t_n) \mapsto e^{-\int_0^{t_n} \lambda(u) du} \left(\prod_{i=1}^n \lambda(t_i) \right) \mathbb{1}_{\{0 < t_1 < \dots < t_n\}}.$$

Note that if λ is a constant function, we recover the density in the continuous case (c.f. Lemma 2.19).

Proof. Admitted here.

Main argument: mathematical induction on n , using the Time change Proposition 4.7. \square

We can thus deduce the conditional distribution of the arrival times.

Proposition 4.11.

Let $n \in \mathbb{N}^*$ and $t \in \mathbb{R}_+^*$. Given that $\{N_t = n\}$, the n arrival times $0 < T_1 < \dots < T_n$ in $[0, t]$ of an inhomogeneous Poisson process with intensity function $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, has conditional density w.r.t. the Lebesgue measure on \mathbb{R}^n :

$$(t_1, \dots, t_n) \mapsto \frac{n!}{\int_0^t \lambda(u) du} \left(\prod_{i=1}^n \lambda(t_i) \right) \mathbb{1}_{\{0 < t_1 < \dots < t_n \leq t\}}.$$

In particular, conditionally on $\{N_t = n\}$, (T_1, \dots, T_n) behaves as the order statistic associated to n i.i.d. r.v. with common density

$$s \in \mathbb{R} \mapsto \frac{\lambda(s)}{\int_0^t \lambda(u) du} \mathbb{1}_{\{0 < s \leq t\}}.$$

Proof. Similar to the one of Proposition 2.23. \square

This result leads to a new construction of an inhomogeneous Poisson process.

Proposition 4.12.

Let $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an intensity function. Let $Y \sim \mathcal{P} \left(\int_0^T \lambda(u) du \right)$.

- If $Y = 0$, then for all $0 \leq t \leq T$, $N_t = 0$ (the point process has no points in $[0, T]$).
- Otherwise, let S_1, \dots, S_Y be i.i.d. r.v. with density w.r.t. the Lebesgue measure on \mathbb{R}

$$s \in \mathbb{R} \mapsto \frac{\lambda(s)}{\int_0^T \lambda(u) du} \mathbb{1}_{\{0 < s \leq T\}}.$$

Sort the values $(T_1, \dots, T_Y) = (S_{(1)}, \dots, S_{(Y)})$ and consider the associated counting process defined for all $0 \leq t \leq T$,

$$N_t = \sum_{n=1}^Y \mathbb{1}_{\{T_n \leq t\}} = \sum_{i=1}^Y \mathbb{1}_{\{S_i \leq t\}}.$$

Then $(N_t)_{t \in [0, T]}$ is an inhomogeneous Poisson process with intensity function λ on $[0, T]$.

Proof. • One the one hand, on the event $\{Y = 0\}$, $N_t = 0$.

- On the other hand, on the event $\{Y = n\}$,

$$N_t = \sum_{i=1}^n \mathbb{1}_{\{S_i \leq t\}}.$$

To prove that $(N_t)_{t \in [0, T]}$ is a Poisson process, we need to prove that for all subdivision of $[0, T]$,

$$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = T,$$

and for all integers k_1, \dots, k_m , one has

$$\mathbb{P}(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m) = \prod_{j=1}^m \left(\frac{c_j^{k_j}}{k_j!} e^{-c_j} \right), \quad (4.1)$$

where

$$c_j = \int_{t_{j-1}}^{t_j} \lambda(u) du = \Lambda(t_j) - \Lambda(t_{j-1}).$$

Yet, since $N_{t_m} = N_T = Y$ (there are Y S_i 's that belong to $[0, T]$),

$$\begin{aligned} & \mathbb{P}(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m) \\ &= \mathbb{P}\left(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m, Y = \sum_{j=1}^m k_j\right) \\ &= \mathbb{P}\left(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m \mid Y = \sum_{j=1}^m k_j\right) \mathbb{P}\left(Y = \sum_{j=1}^m k_j\right) \end{aligned}$$

On the one hand, for all j , for all i , by definition of the S_i 's,

$$\mathbb{P}(S_i \in (t_{j-1}, t_j]) = \int_{t_{j-1}}^{t_j} \frac{\lambda(s)}{\Lambda(T)} ds = \frac{c_j}{c},$$

where c denotes

$$c = \int_0^T \lambda(u) du = c_1 + \dots + c_m = \Lambda(T).$$

Hence, conditionally on $\left\{Y = \sum_{j=1}^m k_j\right\}$, the random variables $\left(\sum_{i=1}^{k_1+\dots+k_m} \mathbb{1}_{\{S_i \in (t_{j-1}, t_j]\}}\right)_{1 \leq j \leq m}$, which count the number of S_i 's in each interval $(t_{j-1}, t_j]$, have a multinomial distribution with parameters $(k_1 + \dots + k_m)$ and $(c_j/c)_{1 \leq j \leq m}$. Therefore,

$$\begin{aligned} & \mathbb{P}\left(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m \mid Y = \sum_{j=1}^m k_j\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{k_1+\dots+k_m} \mathbb{1}_{\{S_i \in (t_0, t_1]\}} = k_1, \dots, \sum_{i=1}^{k_1+\dots+k_m} \mathbb{1}_{\{S_i \in (t_{m-1}, t_m]\}} = k_m\right) \\ &= \frac{(k_1 + k_2 + \dots + k_m)!}{k_1! \dots k_m!} \left(\frac{c_1}{c}\right)^{k_1} \dots \left(\frac{c_m}{c}\right)^{k_m} \end{aligned}$$

On the other hand, $Y \sim \mathcal{P}(\Lambda(T))$, with $\Lambda(T) = c = c_1 + \dots + c_m$.

Finally,

$$\begin{aligned} & \mathbb{P}(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m) \\ &= \frac{(k_1 + k_2 + \dots + k_m)!}{k_1! \dots k_m!} \left(\frac{c_1}{c}\right)^{k_1} \dots \left(\frac{c_m}{c}\right)^{k_m} \frac{c^{\sum_{j=1}^m k_j}}{(k_1 + \dots + k_m)!} e^{-c} \\ &= \frac{c_1^{k_1}}{k_1!} e^{-c_1} \times \dots \times \frac{c_m^{k_m}}{k_m!} e^{-c_m}, \end{aligned}$$

which ends the proof of (4.1). □

IV Statistics for inhomogeneous Poisson processes

As in the homogeneous case, there are two types of observation: either n is fixed (and the observation window is random), or the observation window is fixed (and you observe a random number of events). Let us focus on the fixed window case.

If one has a parametric model $\{\lambda_\theta; \theta \in \Theta\}$, one can introduce the Maximum Likelihood Estimator (MLE). To do so, we need to compute the Likelihood of the unknown parameter θ .

Proposition 4.13.

Let N be an inhomogeneous Poisson process with intensity function $\lambda_\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ depending on an unknown parameter θ in Θ . Consider the number N_T of "events" that have occurred in $[0, T]$, and denote (T_1, \dots, T_{N_T}) the corresponding arrival times. Then likelihood of the trajectory of N is equal to

$$\mathcal{L}((N_t)_{t \in [0, T]}; \theta) = \left(\prod_{n=1}^{N_T} \lambda_\theta(T_n) \right) e^{-\int_0^T \lambda_\theta(u) du}.$$

Then, if it exists, one can compute the MLE of θ ,

$$\hat{\theta}_{MLE} \in \operatorname{argmax}_{\theta \in \Theta} \mathcal{L}((N_t)_{t \in [0, T]}; \theta),$$

and estimate the intensity function by $\lambda_{\hat{\theta}_{MLE}}$.

You will have the opportunity to deepen your understanding of statistical inference for inhomogeneous Poisson processes through various mini-projects.