Chapter 3

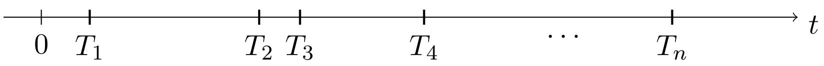
Statistics for homogeneous Poisson processes

In this part, we do some statistical inference for homogeneous Poisson processes. We observe a random point process $N = (N_t)_{t\in\mathbb{R}_+}$ with arrival times $(T_n)_{n\in\mathbb{N}^*}$. We assume that N is a homogeneous Poisson process with *unknown* intensity $\lambda > 0$. We aim at estimating λ. We also aim at constructing (asymptotic) confidence intervals for λ, and statistical tests.

I Introduction

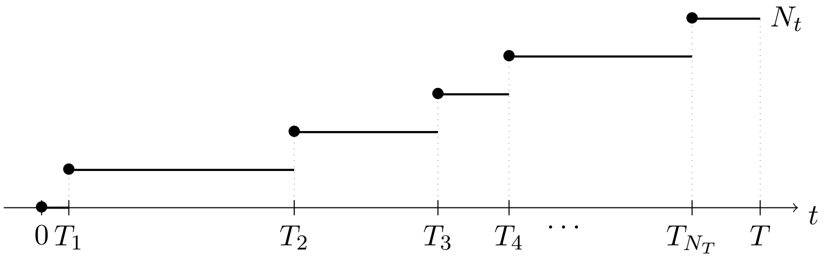
There are two types of observation for a point process:

- Either n is fixed, and we observe the process up to the nth event, that is we observe $(T_1, \dots, T_n)$.



  In this case, the observation interval/window $[0, T_n]$ is random.

- Either the observation interval $[0, T]$ is fixed, and we observe the process on a fixed interval, that is we observe $(N_t)_{t\in[0,T]}$.



  In this case, the number of events occurring in this interval N_T is random.

II Fixed number of points, random observation time

In this section n is fixed, and we observe a homogeneous Poisson process N with (unknown) intensity $\lambda > 0$ up to the nth arrival times. In particular, we observe the n first arrival times $(T_1, \dots, T_n)$ or equivalently the n first interarrival times $(W_1, \dots, W_n)$. Then, according to Proposition 2.18, the $(W_i)_{i\geq 1}$ are i.i.d. with distribution $\mathcal{E}(\lambda)$ which makes the study easier. In particular, T_n is $\Gamma(n, \lambda)$-distributed.

II.1 Maximum Likelihood Estimator (MLE)

Proposition 3.1.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$. Consider the n first arrival times $(T_1, \ldots, T_n)$ of N. Then likelihood of the trajectory of N is equal to

$$\mathcal{L}\left((T_1, \ldots, T_n); \lambda\right) = \lambda^n e^{-\lambda T_n}.$$

Hence, the Maximum Likelihood Estimator (MLE) of λ is

$$\hat{\lambda}_n = \frac{n}{T_n}.$$

Proof. In exercise.

1. Recall the joint density of $(T_1, \ldots, T_n)$, and deduce its likelihood.

Solution.

According to Lemma 2.19, we know that the density of $(T_1, \ldots, T_n)$ is

$$(t_1, \ldots, t_n) \mapsto \lambda^n e^{-\lambda t_n} \mathbb{1}_{\{0<t_1<\ldots<t_n\}}.$$

Hence its likelihood equals

$$\mathcal{L}\left((T_1, \ldots, T_n); \lambda\right) = \lambda^n e^{-\lambda T_n} \mathbb{1}_{\{0<T_1<\ldots<T_n\}},$$

with $\mathbb{1}_{\{0<T_1<\ldots<T_n\}} = 1$ a.s.

2. Compute the MLE of λ.

Solution.

We deduce that the log-likelihood equals

$$\ell\left((T_1, \ldots, T_n); \lambda\right) = n \ln(\lambda) - \lambda T_n.$$

Hence

$$\frac{\partial \ell}{\partial \lambda}\left((T_1, \ldots, T_n); \lambda\right) = \frac{n}{\lambda} - T_n = 0 \quad \Leftrightarrow \quad \lambda = \frac{n}{T_n}.$$

Moreover, n/T_n is a maximum since the log-likelihood function is concave. Indeed,

$$\frac{\partial^2 \ell}{\partial \lambda^2}\left((T_1, \ldots, T_n); \lambda\right) = \frac{-n}{\lambda^2} < 0.$$

□

Note that we could recover this result using that knowing the point process is also equivalent to knowing the interarrival times which are i.i.d. with $\mathcal{E}(\lambda)$ distribution. Indeed,

$$\mathcal{L}\left((W_1, \ldots, W_n); \lambda\right) = \prod_{i=1}^{n} \left(\lambda e^{-\lambda W_i} \mathbb{1}_{\{W_i>0\}}\right) = \lambda^n e^{-\lambda \sum_{i=1}^n W_i} = \lambda^n e^{-\lambda T_n}.$$

II.2 Non-asymptotic properties of the MLE

Lemma 3.2.

Let N be a homogeneous Poisson process with intensity $\lambda > 0$ and $n \geq 2$ a fixed integer. Then, the MLE of λ satisfies

$$\mathbb{E}\left[\hat{\lambda}_n\right] = \frac{n}{n-1}\lambda.$$

In particular, from Theorem 3.5, we can deduce that in the case "fixed n", the Maximum Likelihood Estimator $\hat{\lambda}_n$ of λ is biased, and asymptotically unbiased.

However, we can consider

$$\tilde{\lambda} = \frac{n-1}{T_n},$$

and prove that $\tilde{\lambda}$ is an unbiased estimator of λ, but not efficient.

Proof. Worksheet 3, exercise 1. □

Moreover, to construct statistical tests with uniform asymptotic level, we need the following lemma.

Lemma 3.3.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$ observed up to the nth first arrival time T_n and consider the MLE $\hat{\lambda}_n = n/T_n$ of λ.

Then, for all (fixed) $t > 0$,

$$\lambda \mapsto \mathbb{P}_\lambda\left(\hat{\lambda}_n \geq t\right)$$

is a non-decreasing function.

Proof. In exercise. Consider $\lambda \leq \mu$ and prove that $\mathbb{P}_\lambda\left(\hat{\lambda}_n \geq t\right) \leq \mathbb{P}_\mu\left(\hat{\lambda}_n \geq t\right)$. *Hint:* Use Proposition 1.18.

Solution.

Note that $\hat{\lambda}_n = n/T_n$ with $T_n \sim \Gamma(n, \lambda)$. Hence, by Proposition 1.18,

$$2\lambda T_n \sim \Gamma(n, 1/2) \overset{(d)}{=} \chi^2(2n).$$

Therefore, for $\lambda \leq \mu$,

$$\begin{aligned}
\mathbb{P}_\lambda\left(\hat{\lambda}_n \geq t\right) &= \mathbb{P}_\lambda\left(T_n \leq \frac{n}{t}\right) \\
&= \mathbb{P}_\lambda\left(2\lambda T_n \leq \frac{2\lambda n}{t}\right) \\
&= \mathbb{P}\left(X \leq \frac{2\lambda n}{t}\right) && \text{where } X \sim \chi^2(2n) \\
&\overset{(*)}{\leq} \mathbb{P}\left(X \leq \frac{2\mu n}{t}\right) && \text{since } \lambda \leq \mu \\
&= \mathbb{P}_\mu\left(2\mu T_n \leq \frac{2\mu n}{t}\right) \\
&= \mathbb{P}_\mu\left(\hat{\lambda}_n \geq t\right).
\end{aligned}$$

where $^{(*)}$ holds from the fact that a c.d.f. is (always) a non-decreasing function.

□

II.3 Asymptotic distribution of the MLE

Before studying the asymptotic properties of the MLE, recall the univariate delta method.

Lemma 3.4 (Univariate delta method)**.**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables, Y be a real-valued random variable and θ be in $\mathbb{R}$ such that

$$r_n(X_n-\theta) \xrightarrow[n\to+\infty]{\mathcal{L}} Y,$$

where (r_n) is a sequence of non-negative real numbers that tends toward infinity. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function at θ. Then,

$$f(X_n) \xrightarrow[n\to+\infty]{\mathbb{P}} f(\theta) \qquad \text{and} \qquad r_n\left(f(X_n)-f(\theta)\right) \xrightarrow[n\to+\infty]{\mathcal{L}} f'(\theta)Y.$$

This allows us to deduce the asymptotic behavior of the MLE.

Theorem 3.5.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda>0$ observed up to the nth first arrival time T_n and consider the MLE $\hat{\lambda}_n = n/T_n$ of λ.

1. **LLN-type result:** *$\hat{\lambda}_n$ is a consistent estimator of λ, that is*

$$\frac{n}{T_n} \xrightarrow[n\to+\infty]{\mathbb{P}} \lambda.$$

2. **CLT-type result:** *Moreover, it is asymptotically gaussian, that is*

$$\sqrt{n}\left(\frac{n}{T_n}-\lambda\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0,\lambda^2).$$

Comment 3.6. Note that here, the asymptotic variance equals λ^2.

Proof. In exercise.

1. Apply the CLT to the interarrival times $(W_i)_{i\geq 1}$, and prove that

$$\sqrt{n}\left(\frac{1}{\hat{\lambda}_n}-\frac{1}{\lambda}\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\left(0,\frac{1}{\lambda^2}\right). \tag{3.1}$$

Solution.

Recall that $\hat{\lambda}_n = n/T_n$ where

$$\frac{T_n}{n} = \frac{1}{n}\sum_{i=1}^{n} W_i = \bar{W}_n.$$

Applying the CLT to the sequence $(W_i)_{i\geq 1}$ which are i.i.d. random variables such that

$$\mathbb{E}[|W_1|] = \mathbb{E}[W_1] = 1/\lambda < +\infty \qquad \text{and} \qquad \mathrm{Var}(W_1) = 1/\lambda^2 < +\infty,$$

leads to

$$\sqrt{n}\frac{\left(\frac{1}{\hat{\lambda}_n}-\frac{1}{\lambda}\right)}{\sqrt{1/\lambda^2}} = \sqrt{n}\frac{\left(\bar{W}_n-\frac{1}{\lambda}\right)}{\sqrt{1/\lambda^2}} = \sqrt{n}\frac{(\bar{W}_n-\mathbb{E}[W_1])}{\sqrt{\mathrm{Var}(W_1)}} \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

We deduce (3.1) by multiplying by $\sqrt{1/\lambda^2}$ (which multiplies the variance by $(1/\lambda^2)$).

2. Deduce Theorem 3.5 from the delta method.

Solution.

Let us apply the delta method to the differentiable function

$$f : x \in \mathbb{R}_+^* \mapsto \frac{1}{x}, \qquad \text{with, by (3.1),} \qquad \begin{cases} r_n = \sqrt{n}, \\ X_n = 1/\hat{\lambda}_n, \\ \theta = 1/\lambda, \\ Y \sim \mathcal{N}(0, 1/\lambda^2). \end{cases}$$

(a) Since $f(X_n) = \hat{\lambda}_n$ and $f(\theta) = \lambda$, we first obtain the LLN-type result:

$$\hat{\lambda}_n \xrightarrow[n\to+\infty]{\mathbb{P}} \lambda.$$

(b) Moreover, $\forall x \in \mathbb{R}_+^*$, $f'(x) = -1/x^2$. Hence,

$$f'(\theta)Y = -\lambda^2 Y \quad \sim \quad \mathcal{N}\left(0, (-\lambda^2)^2 \times [1/\lambda^2]\right),$$

and we obtain the CLT-type result:

$$\sqrt{n}\left(\hat{\lambda}_n - \lambda\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, \lambda^2\right).$$

□

II.4 Confidence intervals and statistical tests

Fix n in $\mathbb{N}^*$ and let α in $(0,1)$. We observe the n first arrival times of a homogeneous Poisson process with intensity $\lambda > 0$ and we aim at constructing confidence intervals for λ with confidence level $1-\alpha$, or at testing, for instance,

$$\mathcal{H}_0 : \lambda \leq \lambda_0 \quad \text{against} \quad \mathcal{H}_1 : \lambda > \lambda_0, \qquad \text{at level } \alpha.$$

A) Non-asymptotic statistical inference

Based on Proposition 1.18, one may construct a (non-asymptotic) confidence interval for λ with confidence level $1-\alpha$, that is

$$\left[\frac{x_{2n,\alpha/2}}{2T_n}; \frac{x_{2n,1-\alpha/2}}{2T_n}\right],$$

where $x_{d,\eta}$ denotes the η-quantile of a $\chi^2(d)$ distribution.

Proof. In exercise. □

Equivalently, one may prove that the test which rejects $\mathcal{H}_0$ if $2\lambda_0 T_n \leq x_{2n,\alpha}$ is of (non-asymptotic) level α, that is

$$\sup_{\lambda \leq \lambda_0} \{\mathbb{P}_\lambda(2\lambda_0 T_n \leq x_{2n,\alpha})\} \leq \alpha.$$

The test statistic of this test is

$$X_n := 2\lambda_0 T_n.$$

Moreover, if $\lambda = \lambda_0$, $X_n \sim \chi^2(2n)$. Hence, the p-value of this test equals

$$\mathbb{P}_{\lambda_0}\left(X_n \leq X_n^{obs}\right) = F_{2n}\left(X_n^{obs}\right),$$

where X_n^{obs} is the observed value of the test statistic (numerical application) and F_{2n} denotes the c.d.f. of a $\chi^2(2n)$ distribution.

Proof. Facultative homework. □

B) Asymptotic statistical inference

From Theorem 3.5, one can construct an asymptotic confidence interval that is

$$\left[\frac{n}{T_n\left(1+\frac{z_{1-\alpha/2}}{\sqrt{n}}\right)}; \frac{n}{T_n\left(1-\frac{z_{1-\alpha/2}}{\sqrt{n}}\right)}\right],$$

where $z_{1-\alpha/2}$ denotes the $(1-\alpha/2)$-quantile of a standard gaussian distribution $\mathcal{N}(0,1)$.

Proof. In exercise □

Moreover, one can prove that the test which rejects $\mathcal{H}_0$ if $\hat{\lambda}_n \geq \lambda_0(1 + z_{1-\alpha}/\sqrt{n})$ is (uniformly) asymptotically of level α, that is

$$\limsup_{n\to+\infty}\left(\sup_{\lambda\leq\lambda_0}\left\{\mathbb{P}_\lambda\left(\hat{\lambda}_n \geq \lambda_0\left(1+\frac{z_{1-\alpha}}{\sqrt{n}}\right)\right)\right\}\right) \leq \alpha.$$

This result lies on Lemma 3.3.

Note that, by isolating the quantile in the rejection rule, we recover the test statistic, that is

$$Z_n := \sqrt{n}\left(\frac{\hat{\lambda}_n}{\lambda_0} - 1\right) = \sqrt{n}\left(\frac{n}{\lambda_0 T_n} - 1\right).$$

Moreover, if $\lambda = \lambda_0$, $Z_n \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0,1)$. Hence the p-value of this test equals

$$\mathbb{P}_{\lambda_0}\left(Z_n \geq Z_n^{obs}\right) \approx 1 - \Phi\left(Z_n^{obs}\right),$$

where Z_n^{obs} denotes the observed value of the test statistic (numerical application), and Φ denotes the c.d.f. of a standard gaussian distribution $\mathcal{N}(0,1)$.

Proof. Facultative homework. □

III Fixed window, random number of points

In this section T is fixed, and we observe a homogeneous Poisson process N with (unknown) intensity $\lambda > 0$ on the interval $[0, T]$. In particular, the number of "events" that have occurred by time T, that is N_T, is random.

III.1 Maximum Likelihood Estimator

Heuristic: Recall that, if X is a random variable with density f_θ and c.d.f. F_θ. Then, its likelihood in θ equals

$$\mathcal{L}(X, \theta) = f_\theta(X),$$

where,

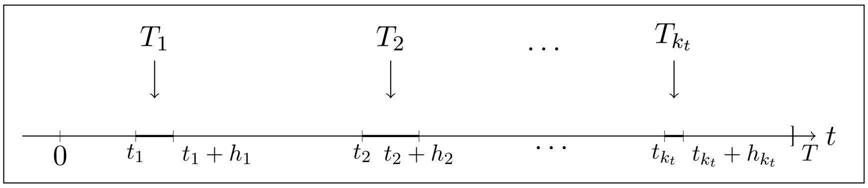
$$f_\theta(x) = F'_\theta(x) = \lim_{h \to 0} \frac{F_\theta(x+h) - F_\theta(x)}{h} = \lim_{h \to 0} \frac{\mathbb{P}_\theta(X \in (t; t+h])}{h}.$$

In the point process case: Recall that in the "fixed-T" case, $N_T = \sum_{i=1}^{+\infty} \mathbb{1}_{\{T_i \le T\}}$ is random. Similarly to the proof of Lemma 2.19, consider a strictly increasing sequence $(t_i)_{i \in \mathbb{N}^*}$ in $\mathbb{R}_+$, and denote

$$k_t = \sum_{i=1}^{+\infty} \mathbb{1}_{\{t_i \le T\}} \qquad \text{s.t.} \qquad 0 =: t_0 < t_1 < \ldots < t_{k_t} < T < t_{k_t+1} < \ldots.$$

Hence, k_t counts the number of t_i's that belong to $[0, T]$.
Introduce for $h = (h_1, \ldots, h_n)$ small enough such that all $(t_i, t_i + h_i]$ are disjoint (and $t_k + h_k \le T$),

$$B_h = \{1 \text{ point in each } (t_i, t_i + h_i] \text{ and } 0 \text{ points elsewhere in } [0, T]\}.$$



The main difference with A_h in Lemma 2.19 is that now n in not fixed anymore, and we need to consider k_t which is the number of (t_i) which belong to the observation interval $[0, T]$.

Heuristically, as in the proof of Lemma 2.19,

$$f((t_i)_{i \in \mathbb{N}^*} \cap [0, T]) = \lim_{\forall i,\ h_i \to 0} \frac{\mathbb{P}(B_h)}{h_1 \ldots h_{k_t}}.$$

Since the intervals appearing in B_h are disjoints, by independence of the increments,

$$\begin{aligned} \mathbb{P}(B_h) &= \left[\prod_{i=1}^{k_t} \mathbb{P}(1 \text{ point in } (t_i, t_i + h_i]) \right] \times \mathbb{P}(0 \text{ points elsewhere}) \\ &\overset{(*)}{=} \left[\prod_{i=1}^{k_t} \lambda h_i e^{-\lambda h_i} \right] \times e^{-\lambda (T - \sum_{i=1}^{k_t} h_i)} \\ &= \lambda^{k_t} h_1 \ldots h_{k_t} e^{-\lambda T}. \end{aligned}$$

with $^{(*)}$ is because "elsewhere" is of length $T - \sum_{i=1}^{k_t} h_i$.

Finally,

$$f((t_i)_{i \in \mathbb{N}^*} \cap [0, T]) = \lim_{\forall i,\ h_i \to 0} \lambda^{k_t} e^{-\lambda T} = \lambda^{k_t} e^{-\lambda T}.$$

Proposition 3.7.

Let N be a homogeneous Poisson process with (unknown) intensity λ. Consider the number N_T of "events" that have occurred in $[0,T]$, dans denote $(T_1,\dots,T_{N_T})$ the corresponding arrival times. Then likelihood of the trajectory of N is equal to

$$\mathcal{L}\left((N_t)_{t\in[0,T]};\lambda\right) = \lambda^{N_T}e^{-\lambda T}.$$

Hence, the Maximum Likelihood Estimator (MLE) of λ is

$$\hat{\lambda}_T = \frac{N_T}{T}.$$

Notice that the likelihood can be expressed as follows:

$$\mathcal{L}\left((N_t)_{t\in[0,T]};\lambda\right) = \left[N_T!\left(\prod_{i=1}^{N_T}\frac{1}{T}\mathbb{1}_{\{T_i\in[0,T]\}}\right)\mathbb{1}_{\{T_1<\ldots<T_{N_T}\}}\right]\times\left[\frac{(\lambda T)^{N_T}e^{-\lambda T}}{N_T!}\right].$$

We recognize the product of

- the (conditional) likelihood of $(T_1,\dots,T_{N_T})|N_T$, that is, according to Proposition 2.23, the likelihood of the order statistic corresponding to N_T independent random variables uniformly distributed on the interval $[0,T]$ given N_T,
- and the likelihood of N_T which has a Poisson distribution $\mathcal{P}(\lambda T)$.

Proof. In exercise. Compute the MLE of λ.
The expression of the likelihood is admitted here (c.f. heuristic above).

Solution.

We deduce that the log-likelihood equals

$$\ell\left((N_t)_{t\in[0;T]};\lambda\right) = N_T\ln(\lambda) - \lambda T.$$

Hence

$$\frac{\partial\ell}{\partial\lambda}\left((N_t)_{t\in[0;T]};\lambda\right) = \frac{N_T}{\lambda} - T = 0 \qquad \Leftrightarrow \qquad \lambda = \frac{N_T}{T}.$$

Moreover, N_T/T is a maximum since the log-likelihood function is concave. Indeed,

$$\frac{\partial^2\ell}{\partial\lambda^2}\left((N_t)_{t\in[0;T]};\lambda\right) = \frac{-N_T}{\lambda^2} \le 0 \quad \text{a.s.}$$

□

III.2 Non-asymptotic properties of the MLE

Proposition 3.8.

Let $(N_t)_{t\in[0,T]}$ be a Poisson process with rate $\lambda>0$ observed on $[0,T]$ and consider the MLE $\hat{\lambda}_T = N_T/T$. Since $N_T\sim\mathcal{P}(\lambda T)$,

$$\mathbb{E}\left[\hat{\lambda}_T\right] = \lambda \qquad \text{and} \qquad \mathrm{Var}\left(\hat{\lambda}_T\right) = \frac{\lambda}{T}.$$

We deduce that $\hat{\lambda}_T$ is an unbiased and efficient estimator of λ.

Proof. In exercise.

Solution.

Since $N_T \sim \mathcal{P}(\lambda T)$, $\mathbb{E}[N_T] = \lambda T$ and $\mathrm{Var}(N_T) = \lambda T$. Hence, on the one hand, by linearity of the expectation,

$$\mathbb{E}\left[\hat{\lambda}_T\right] = \frac{\mathbb{E}[N_T]}{T} = \lambda.$$

We deduce that $\hat{\lambda}_T$ is an unbiased estimator of λ. On the other hand, by property of the variance,

$$\mathrm{Var}\left(\hat{\lambda}_T\right) = \frac{\mathbb{E}[N_T]}{T^2} = \frac{\lambda}{T}.$$

Moreover, one can compute the Fisher information:

$$I_T(\lambda) = \mathbb{E}_\lambda\left[-\frac{\partial^2}{\partial\lambda^2}\ell\left((N_t)_{t\in[0,T]};\lambda\right)\right] = \mathbb{E}_\lambda\left[\frac{N_T}{\lambda^2}\right] = \frac{T}{\lambda}.$$

Thus the Cramér-Rao bound equals

$$\mathcal{B}_T(\lambda) = \frac{1}{I_T(\lambda)} = \frac{\lambda}{T} = \mathrm{Var}\left(\hat{\lambda}_T\right).$$

We deduce that $\hat{\lambda}_T$ is efficient.

□

Moreover, to construct statistical tests with uniform asymptotic level, we need the following lemma.

Lemma 3.9.

Let N be a homogeneous Poisson process with (unknown) intensity $\lambda > 0$ observed on the interval $[0, T]$ and consider the MLE $\hat{\lambda}_T = N_T/T$ of λ.
Then, for all (fixed) $x \geq 0$,

$$\lambda \mapsto \mathbb{P}_\lambda\left(\hat{\lambda}_T \geq x\right)$$

is a non-decreasing function on $\mathbb{R}_+^*$.

Proof. In exercise. Fix $x \geq 0$ and denote

$$g : \lambda > 0 \mapsto \mathbb{P}_\lambda\left(\hat{\lambda}_T \geq x\right).$$

1. Assume $x = 0$. Compute $g(\lambda)$ for all $\lambda > 0$ and conclude in this case.

Solution.

If $x = 0$, g is constant equal to 1, since for all $\lambda > 0$, $\mathbb{P}\left(\hat{\lambda}_T \geq 0\right) = \mathbb{P}(N_T \geq 0) = 1$.

2. Now assume $x > 0$. Justify that for all $\lambda > 0$,

$$g(\lambda) = \sum_{k=\lceil xT \rceil}^{+\infty} \frac{(\lambda T)^k}{k!} e^{-\lambda T},$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

Solution.

Note that

$$g(\lambda) = \mathbb{P}\left(\hat{\lambda}_T \geq x\right) = \mathbb{P}(N_T \geq xT) = \mathbb{P}(N_T \geq \lceil xT \rceil),$$

since N_T is an integer. The expression of $g(\lambda)$ is then immediate since $N_T \sim \mathcal{P}(\lambda T)$.

3. Deduce that

$$g'(\lambda) = \frac{(\lambda T)^{\lceil xT \rceil - 1}}{(\lceil xT \rceil - 1)!} T e^{-\lambda T}.$$

Hint: Use the method of differences, that is telescoping series.

Solution.

We obtain

$$\begin{aligned} g'(\lambda) &= \left[\sum_{k=\lceil xT \rceil}^{+\infty} \frac{\lambda^{k-1} T^k}{(k-1)!} e^{-\lambda T} \right] - \left[\sum_{k=\lceil xT \rceil}^{+\infty} \frac{(\lambda T)^k}{k!} T e^{-\lambda T} \right] \\ &\overset{(*)}{=} \left(\sum_{k=\lceil xT \rceil - 1}^{+\infty} \frac{(\lambda T)^k}{k!} - \sum_{k=\lceil xT \rceil}^{+\infty} \frac{(\lambda T)^k}{k!} \right) T e^{-\lambda T} \\ &\overset{(\dagger)}{=} \frac{(\lambda T)^{\lceil xT \rceil - 1}}{(\lceil xT \rceil - 1)!} T e^{-\lambda T}, \end{aligned}$$

with $^{(*)}$ by change of variable $k \leftarrow k - 1$ (since $\lceil xT \rceil \geq 1$), and $^{(\dagger)}$ by the difference method.

4. Conclude.

Solution.

We deduce that $g'(\lambda) > 0$ for all $\lambda > 0$, that is g is (strictly) increasing when $x > 0$.

□

III.3 Asymptotic distribution of the MLE

Theorem 3.10.

Let $(N_t)_{t \in [0,T]}$ be a Poisson process with rate $\lambda > 0$ observed on $[0, T]$ and consider the MLE $\hat{\lambda}_T = N_T / T$.

1. *LLN-type result: $\hat{\lambda}_T$ is a consistent estimator of λ, that is*

$$\frac{N_T}{T} \xrightarrow[T \to +\infty]{\mathbb{P}} \lambda.$$

2. *CLT-type result: Moreover, it is asymptotically gaussian, that is*

$$\sqrt{T} \left(\frac{N_T}{T} - \lambda \right) \xrightarrow[T \to +\infty]{\mathcal{L}} \mathcal{N}(0, \lambda).$$

Note that it is also possible to prove that $N_T \xrightarrow[T \to +\infty]{a.s.} +\infty$ and deduce the "strong" version of the LLN-type result:

$$\frac{N_T}{T} \xrightarrow[T \to +\infty]{a.s.} \lambda.$$

This result is based on the following inequality: $\frac{T_{N_T}}{N_T} \leq \frac{T}{N_T} < \frac{T_{N_T+1}}{N_T}$.

Proof. In exercise.

1. Note that we cannot apply the classical Law of Large Numbers since the index in the limit is not a countable sequence. Apply Chebychev's inequality to prove the consistency of $\hat{\lambda}_T$.

Solution.

Since $\mathbb{E}\left[\hat{\lambda}_T\right] = \lambda$ and $\mathrm{Var}\left(\hat{\lambda}_T\right) = \lambda/T < +\infty$, for all $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\hat{\lambda}_T - \lambda\right| \geq \varepsilon\right) \quad \leq \quad \frac{\mathrm{Var}\left(\hat{\lambda}_T\right)}{\varepsilon^2} = \frac{\lambda}{T\varepsilon^2} \quad \underset{T\to+\infty}{\longrightarrow} \quad 0.$$

2. Prove that the characteristic function of $Z_T := \sqrt{T}(N_t/T - \lambda)$ equals for all $u \in \mathbb{R}$,

$$\mathbb{E}\left[e^{iuZ_T}\right] = \exp\left(\lambda T\left[e^{\frac{iu}{\sqrt{T}}} - 1 - \frac{iu}{\sqrt{T}}\right]\right).$$

Hint: Use Proposition 1.2.

Solution.

Let $u \in \mathbb{R}$.

$$\mathbb{E}\left[e^{iuZ_T}\right] = \mathbb{E}\left[e^{iu\sqrt{T}\left(\frac{N_T}{T}-\lambda\right)}\right] = \mathbb{E}\left[e^{i\frac{u}{\sqrt{T}}N_T}\right]e^{-iu\lambda\sqrt{T}}$$

Yet $N_T \sim \mathcal{P}(\lambda T)$. Hence, by Proposition 1.2, its characteristic function computed in $u/\sqrt{T}$ equals

$$\mathbb{E}\left[e^{i\frac{u}{\sqrt{T}}N_T}\right] = \exp\left(\lambda T\left[e^{i\frac{u}{\sqrt{T}}} - 1\right]\right).$$

Hence, we directly obtain

$$\mathbb{E}\left[e^{iuZ_T}\right] = \exp\left(\lambda T\left[e^{\frac{iu}{\sqrt{T}}} - 1 - \frac{iu}{\sqrt{T}}\right]\right).$$

3. Prove that

$$\lim_{T\to+\infty} \mathbb{E}\left[e^{iuZ_T}\right] = e^{-\frac{\lambda u^2}{2}}.$$

Hint: Use Taylor's expansion.

Solution.

By Taylor's expansion, as $x \to 0$,

$$\exp(x) = 1 + x + \frac{x^2}{2} + o\left(x^2\right).$$

Hence as $T \to +\infty$,

$$\mathbb{E}\left[e^{iuZ_T}\right] = \exp\left(\lambda T\left[\frac{-u^2}{2T} + o\left(\frac{1}{T}\right)\right]\right) = \exp\left(-\frac{\lambda u^2}{2} + o(1)\right) \underset{T\to+\infty}{\longrightarrow} \exp\left(-\frac{\lambda u^2}{2}\right).$$

4. Deduce the asymptotic distribution of Z_T. *Hint:* Recall that the characteristic function of a gaussian r.v. Z with distribution $\mathcal{N}(m,\sigma^2)$ equals for all $u \in \mathbb{R}$, $\mathbb{E}[iuZ] = \exp\left(imu - \frac{u^2\sigma^2}{2}\right)$.

Solution.

We recognize the characteristic function of a $\mathcal{N}(0,\lambda)$, which ends the proof of Theorem 3.10.

□

III.4 Confidence intervals and statistical tests

Fix $T > 0$ and let α in $(0,1)$. We observe a homogeneous Poisson process with unknown intensity $\lambda > 0$ on $[0,T]$ and we aim at constructing confidence intervals for λ with confidence level $1-\alpha$, or at testing, for instance,

$$\mathcal{H}_0 : \lambda \leq \lambda_0 \quad \text{against} \quad \mathcal{H}_1 : \lambda > \lambda_0, \qquad \text{at level } \alpha.$$

From Theorem 3.10 and Slutsky's Lemma, for all α in $(0,1)$, we construct an asymptotic confidence interval for λ with confidence level $1-\alpha$, that is

$$\left[\frac{N_T}{T} - \frac{\sqrt{N_T}}{T} z_{1-\alpha/2}; \frac{N_T}{T} + \frac{\sqrt{N_T}}{T} z_{1-\alpha/2}\right],$$

where $z_{1-\alpha/2}$ denotes the $(1-\alpha/2)$-quantile of a standard gaussian distribution $\mathcal{N}(0,1)$.

Proof. In exercise. □

Moreover, one can construct different test.

Test 1: Reject $\mathcal{H}_0$ if $\hat{\lambda}_T \geq \lambda_0 + \sqrt{\frac{\lambda_0}{T}} z_{1-\alpha}$, with $\begin{cases} \text{test statistic: } Z_T^{(1)} = \sqrt{T}\dfrac{\hat{\lambda}_T - \lambda_0}{\sqrt{\lambda_0}}, \\ p\text{-value: } 1 - \Phi(Z_T^{(1),obs}), \end{cases}$

Test 2: Reject $\mathcal{H}_0$ if $\hat{\lambda}_T \geq \lambda_0 + \sqrt{\frac{\hat{\lambda}_T}{T}} z_{1-\alpha}$, with $\begin{cases} \text{test statistic: } Z_T^{(2)} = \sqrt{T}\dfrac{\hat{\lambda}_T - \lambda_0}{\sqrt{\hat{\lambda}_T}}, \\ p\text{-value: } 1 - \Phi(Z_T^{(2),obs}), \end{cases}$

where $z_{1-\alpha}$ and Φ respectively denote the $(1-\alpha)$-quantile and the c.d.f. of a standard gaussian distribution $\mathcal{N}(0,1)$.

One can prove that both Test 1 and Test 2 are (uniformly) asymptotically of level α, that is

$$\limsup_{n\to+\infty} \left(\sup_{\lambda \leq \lambda_0} \{\mathbb{P}_\lambda(\text{reject } \mathcal{H}_0)\} \right) \leq \alpha.$$

This results relies on Lemma 3.9.

Proof. Facultative homework. □

Take home message

Estimator	Unbiased	Efficient	Asymptotically unbiased	Consistent	Asymptotically gaussian
$\hat{\lambda}_n = \dfrac{n}{T_n}$	✗	·	✓	✓	$\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0, \lambda^2)$
$\tilde{\lambda}_n = \dfrac{n-1}{T_n}$	✓	✗	✓	✓	(?)
$\hat{\lambda}_T = \dfrac{N_T}{T}$	✓	✓	✓	✓	$\sqrt{T}(\hat{\lambda}_T - \lambda) \xrightarrow[T\to+\infty]{\mathcal{L}} \mathcal{N}(0, \lambda)$