Worksheet 3 - Statistics for homogeneous Poisson process

Exercise 1. Unbiased estimation for a fixed number of observations.
In this exercice, we assume that n is fixed, and we observe a homogeneous Poisson process with rate $\lambda > 0$, up to the nth event. Denote $0 < T_1 < T_2 < \ldots < T_n$ the corresponding arrival times.

We estimate the unknown parameter λ by the maximum likelihood estimator (MLE) $\hat{\lambda}_n = n/T_n$.

1. Prove that the MLE is a biased estimator of the rate λ.
2. Consider $\tilde{\lambda}_n = (n-1)/T_n$.
   (a) Check that $\tilde{\lambda}_n$ is an unbiased estimator of λ.
   (b) Is it efficient?

Exercise 2. Quality control.
A manufacturing plant wants to check the quality of its production. To do so, the person in charge keeps count of the number of defective parts.

He thus denotes N_t the number of defective parts manufactured by time t (for any t in hours) and assumes $(N_t)_{t \in \mathbb{R}_+}$ is a Poisson process with unknown intensity $\lambda > 0$. He believes that the quality of the production is not acceptable if the average number of defective parts per hour is greater than or equal to 5.

1. (a) Do you agree with the Poisson process assumption?
   (b) We aim at testing
   $$\mathcal{H}_0 : \lambda \geq \lambda_0 \quad \text{against} \quad \mathcal{H}_1 : \lambda < \lambda_0.$$
   Explicit the value of λ_0 and justify the choice of these testing hypotheses.
2. The person in charge fixes the inspection time to one day and observes $(N_t)_{t \in [0,24]}$. By the end of the day, he counts 108 defective parts.
   (a) Determine an estimation of the rate λ.
   (b) According to you, is the production quality satisfactory?
3. Now, the person in charge decides to observe up to the hundredth flaw and notices that it appears after 22 hours and 15 minutes of production.
   (a) What is an estimation of λ in this case?
   (b) First, we may consider that 100 points is large enough to work with the asymptotic distribution. Given this new observation, is the production quality satisfactory?
   (c) Second, we do not consider the asymptotic distribution anymore. What is your conclusion in this case?

Let us recall the quantiles of the standard gaussian distribution $\mathcal{N}(0,1)$ and the η-quantiles $x_{d,\eta}$ of the $\chi^2(d)$ distribution.

$$\text{If } Z \sim \mathcal{N}(0,1), \quad \begin{cases} \mathbb{P}(Z > 1.645) = 0.95 \\ \mathbb{P}(Z > 1.96) = 0.975 \end{cases}$$

and

$x_{d,\eta}$	$\eta = 0.05$	$\eta = 0.95$
$d = 50$	34.764	67.505
$d = 100$	77.929	124.342
$d = 200$	168.279	233.994