Chapter 4

Inhomogeneous Poisson processes

In this chapter, we consider inhomogeneous Poisson processes. They generalize homogeneous Poisson processes by allowing the rate of the process to be a varying function of time. In particular, by doing so, we drop the stationary increments property.

I Definitions and basic properties

Definition 4.1.

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a real valued piecewise continuous (or at least locally integrable) function defined on $\mathbb{R}_+$. A point process $(N_t)_{t\in\mathbb{R}_+}$ is said to be an inhomogeneous Poisson process with intensity function λ if

(1) $N_0 = 0$.

(2) The process has independent increments.

(3) For all $t > 0$, the number of points in $[0,t]$ has a Poisson distribution with parameter $\int_0^t \lambda(u)du$, that is

$$N_t \sim \mathcal{P}\left(\int_0^t \lambda(u)du\right).$$

The function $\Lambda : t \mapsto \int_0^t \lambda(u)du$ is called *cumulative intensity* of the Poisson process.

Proposition 4.2.

Let N be an inhomogeneous Poisson process with intensity function λ.
Denote for all Borel subset A of $\mathbb{R}_+$, $N(A)$ the number of points in A. Note that if $(T_i)_{i\in\mathbb{N}^*}$ denote the arrival times, then,

$$N(A) = \sum_{n\in\mathbb{N}^*} \mathbb{1}_{\{T_n\in A\}}$$

1. *For all bounded Borel set A in $\mathbb{R}_+$, $N(A)$ is a.s. finite.*

2. *For all $s \geq 0$ and all $t > 0$, the random variable $N(]s, s+t]) = N_{s+t} - N_s$ has a Poisson distribution with parameter $\mu(]s, s+t]) = \int_s^{s+t} \lambda(u)du$, that is*

$$N_{s+t} - N_s \sim \mathcal{P}\left(\int_s^{s+t} \lambda(u)du\right).$$

Proof. 1. Let A be a bounded Borel set in $\mathbb{R}_+$. Hence, there exists $t_A \in \mathbb{R}_+^*$ s.t. $A \subset [0, t_A]$. Therefore,

$$N(A) = \sum_{i \in \mathbb{N}^*} \mathbb{1}_{\{T_n \in A\}} \quad \leq \quad \sum_{i \in \mathbb{N}^*} \mathbb{1}_{\{T_n \leq t_A\}} = N_{t_A}.$$

Moreover, $N_{t_A} \sim \mathcal{P}(\int_0^{t_A} \lambda(u)du)$. Hence $\mathbb{E}[N_{t_A}] = \int_0^{t_A} \lambda(u)du < +\infty$. We deduce that, N_{t_A} is finite a.s., and thus, so is $N(A)$.

2. In exercise:

(a) Recall the moment generating function (or m.g.f.), or Laplace transform, of a Poisson r.v., and for all $t > 0$, express the m.g.f. of N_t w.r.t. the cumulative intensity Λ of the process.

Solution.

Let $X \sim \mathcal{P}(\mu)$. Then for all $u < 0$, $\mathbb{E}\left[e^{uX}\right] = \exp\left(\lambda(e^u - 1)\right)$.
Hence, for any $t > 0$, since $N_t \sim \mathcal{P}(\Lambda(t))$, then for all $u < 0$,

$$\mathbb{E}\left[e^{uN_t}\right] = \exp\left(\Lambda(t)\left[e^u - 1\right]\right).$$

(b) Let $s \geq 0$ and $t > 0$. Compute the m.g.f. of $N_{s+t} - N_s$, and conclude. *Hint:* Use $N_{s+t} = N_{s+t} - N_s + N_s$.

Solution.

Note that $N_{s+t} = N_{s+t} - N_s + N_s$, with $(N_{s+t} - N_s) \perp\!\!\!\perp N_s$ since the intervals $[0, s]$ and $(s, s+t]$ are disjoint. Hence

$$\mathbb{E}\left[e^{uN_{s+t}}\right] = \mathbb{E}\left[e^{u(N_{s+t}-N_s)}e^{uN_s}\right] = \mathbb{E}\left[e^{u(N_{s+t}-N_s)}\right]\mathbb{E}\left[e^{uN_s}\right].$$

Therefore, by the previous question,

$$\exp\left(\Lambda(t+s)\left[e^u - 1\right]\right) = \mathbb{E}\left[e^{u(N_{s+t}-N_s)}\right]\exp\left(\Lambda(s)\left[e^u - 1\right]\right),$$

i.e.

$$\mathbb{E}\left[e^{u(N_{s+t}-N_s)}\right] = \exp\left(\left[\Lambda(s+t) - \Lambda(s)\right]\left[e^u - 1\right]\right) = \exp\left(\int_s^{s+t} \lambda(x)dx\left[e^u - 1\right]\right).$$

We recognize the m.g.f. of a $\mathcal{P}\left(\int_s^{s+t} \lambda(u)du\right)$, which ends the proof.

□

Let us notice several points.

a) If for all $t \geq 0$, $\lambda(t) = \lambda$, that is λ is a constant function, $\int_s^{s+t} \lambda du = \lambda t$, and one recovers the definition of a homogeneous Poisson process with rate λ.

b) Notice that Definition 4.1 characterizes the distribution of the process N. Indeed, for any n and for any $0 \leq t_1 < t_2 < \ldots < t_n$, Definition 4.1 and Proposition 4.2 both provide the distribution of $(N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}})$. Hence, the distribution of the process $(N_t)_{t \in \mathbb{R}+}$ is well known.

c) The terminology of point process theory has been criticized for being too varied. In addition to the word point often being omitted, the homogeneous Poisson (point) process is also called a stationary Poisson (point) process, as well as uniform Poisson (point) process. The inhomogeneous Poisson (point) process, as well as being called nonhomogeneous, is also referred to as the non-stationary Poisson process.

d) The positive measure μ defined on any Borel set A by $\mu(A) = \int_A \lambda(u)du$ (or equivalently $d\mu(u) = \lambda(u)du$) is called the *intensity measure* of the Poisson process N. This definition of an inhomogeneous Poisson process can be generalized to any non-negative intensity measure on $\mathbb{R}_+$ that is locally bounded (i.e. for all bounded Borel subset A in $\mathbb{R}_+$, $\mu(A) < +\infty$), not necessarily absolutely continuous with respect to the Lebesgue measure by replacing assumption (3) by for all Borel set A in $\mathbb{R}_+$, $N(A) \sim \mathcal{P}(\int_A d\mu)$.

However, in this lecture, we restrict attention to the case $d\mu(u) = \lambda(u)du$, that is Poisson processes with intensity functions.

Property 4.3.

Let N be a homogeneous Poisson process with intensity function λ on $\mathbb{R}_+$.

1. The mean function of N equals

$$\forall t \geq 0, \quad m(t) = \mathbb{E}[N_t] = \int_0^t \lambda(u)du = \Lambda(t).$$

   In particular, the rate (see Definition 2.7) of the Poisson process N equals:

$$\forall t \geq 0, \quad w(t) = m'(t) = \lambda(t).$$

2. Moreover, for all $t \geq 0$, as $h \to 0$,
   (a) $\mathbb{P}(N_{t+h} - N_t = 0) = 1 - \lambda(t)h + o(h)$.
   (b) $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda(t)h + o(h)$.
   (c) $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$.

Proof. In exercise.

1. Immediate.
2. To do. *Hint:* Since $\Lambda'(t) = \lambda(t)$, one has $\Lambda(t+h) - \Lambda(t) = \lambda(t)h + o(h)$.

Solution.

By Property 4.3, $N_{t+h} - N_t \sim \mathcal{P}(\Lambda(t+h) - \Lambda(t))$. Then,

(a) $\mathbb{P}(N_{t+h} - N_t = 0) = e^{-[\Lambda(t+h)-\Lambda(t)]} = e^{-\lambda(t)h+o(h)} = 1 - \lambda(t)h + o(h)$.

(b)

$$\begin{aligned}\mathbb{P}(N_{t+h} - N_t = 1) &= [\Lambda(t+h) - \Lambda(t)]\, e^{-[\Lambda(t+h)-\Lambda(t)]} \\ &= [\lambda(t)h + o(h)] \times [1 - \lambda(t)h + o(h)] \\ &= \lambda(t)h + o(h).\end{aligned}$$

(c) $\mathbb{P}(N_{t+h} - N_t \geq 2) = 1 - [\mathbb{P}(N_{t+h} - N_t = 0) + \mathbb{P}(N_{t+h} - N_t = 1)]$.

□

As in the homogeneous case, one can provide an equivalent definition for inhomogeneous Poisson processes.

Definition 4.4.

The point process N is said to be an inhomogeneous Poisson process with intensity function λ defined on if

(1) $N_0 = 0$.

(2) The process has independent increments.

(3) For all positive t, one has as $h \to 0$,

  (a) $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda(t)h + o(h)$.
  (b) $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$.

The proof is the same as in the homogeneous case, except that we obtain a linear differential equation with non-constant coefficients.

Notice that, a Poisson process N with intensity function λ is regular since, for all $t > 0$,

$$\lim_{h \to 0} \frac{\mathbb{P}(N_{t+h} - N_t \geq 2)}{h} = 0.$$

In particular, it is a simple process, that is no more than 1 event can occur at the same time, and each jump of the counting process equals 1 (almost surely). Notice that it is not the case anymore when we generalize to inhomogeneous Poisson processes with non-diffuse intensity measure (i.e. that contains atoms).

II Construction of an inhomogeneous Poisson process

II.1 Time change

A first way of constructing an inhomogeneous Poisson process with intensity function λ on $\mathbb{R}_+$ (and thus proving that such processes exist) is by changing the time scale of a homogeneous Poison process. Before, let us recall the definition and some properties of the generalized inverse function.

A) Generalized inverse function

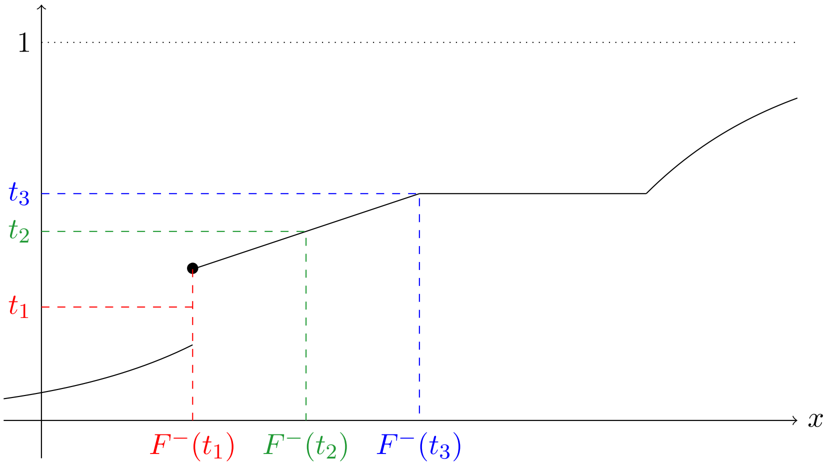
Definition 4.5.

Let $F : \mathbb{R} \to \mathbb{R}_+$ be a non-decreasing càdlàg non-negative function defined on $\mathbb{R}$. Its generalized inverse function is defined by

$$F^- : \begin{pmatrix} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \cup \{\pm\infty\} \\ t & \longmapsto & \inf\{x \in \mathbb{R} \; ; \; F(x) \geq t\} \end{pmatrix},$$

with the conventions $\inf(\varnothing) = +\infty$ and $\inf(\mathbb{R}) = -\infty$.

Note that if F is a c.d.f., then F^- is the corresponding quantile function.



Property 4.6.

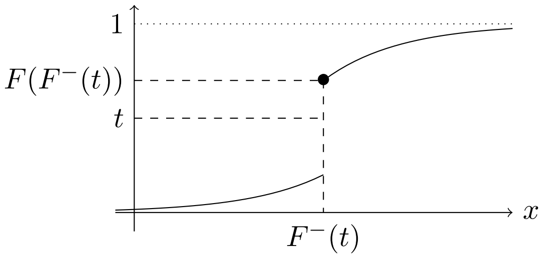
The generalized inverse function satisfies the following properties.

i. F^- is a non-decreasing function.

ii. $F(x) \geq t \quad \Leftrightarrow \quad x \geq F^-(t)$.

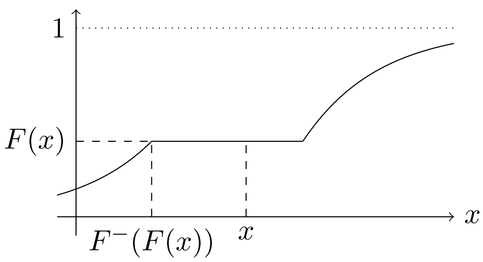
iii. F^- is left-continuous, that is for all t_0 in $\mathbb{R}$, $\lim\limits_{\substack{t \to t_0 \\ t < t_0}} F^-(t) = F(t_0)$.

iv. For all $t \in \mathbb{R}_+$, $F(F^-(t)) \geq t$.



Moreover, if F is continuous, for all $t \in \mathbb{R}_+$, $F(F^-(t)) = t$.

v. For all $x \in \mathbb{R}$, $F^-(F(x)) \leq x$.



Moreover, if F is strictly increasing, for all $x \in \mathbb{R}$, $F^-(F(x)) = x$.

vi. In particular, if F is strictly increasing and continuous, then F is bijective with inverse function $F^{-1} = F^-$.

Proof. Denote for all $t \in \mathbb{R}^+$,

$$A_t = \{x \in \mathbb{R}\ ;\ F(x) \geq t\},$$

such that $F^-(t) = \inf(A_t)$.

i. If $s \leq t$, then, for all $x \in \mathbb{R}$, $F(x) \geq t \Rightarrow F(x) \geq s$. Hence, $A_t \subset A_s$ and thus,

$$F^-(t) = \inf(A_t) \geq \inf(A_s) = F^-(s).$$

ii. $\boxed{\Rightarrow}$ If $F(x) \geq t$, then $x \in A_t$ and thus $F^-(t) = \inf(A_t) \leq x$.

$\boxed{\Leftarrow}$ Assume $x \geq F^-(t)$. Let $\varepsilon > 0$. Then $x + \varepsilon > x \geq F^-(t)$, so $x + \varepsilon > \inf(A_t)$. In particular, there exists $x_\varepsilon \in A_t$ such that $x_\varepsilon \leq x + \varepsilon$. Since F is non-decreasing, then $F(x+\varepsilon) \geq F(x_\varepsilon) \geq t$, as $x_\varepsilon \in A_t$.

We proved that, for all $\varepsilon > 0$, $F(x+\varepsilon) \geq t$. Hence, since F is right-continuous, as $\varepsilon \to 0^+$,

$$F(x) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} F(x+\varepsilon) \geq t.$$

iii. Admitted here. Note that F^- is not right-continuous as soon as F is not strictly increasing.

iv. By applying ii. to $x = F^-(t)$ directly leads to $F(F^-(t)) = F(x) \geq t$.

Assume now that F is continuous. Consider $(x_n)_n$ a sequence satisfying

$$\forall n, \quad x_n < F^-(t) \qquad \text{and} \qquad \lim_{n \to +\infty} x_n = F^-(t).$$

Then by contraposition of ii., for all n, $F(x_n) < t$, and by taking $n \to +\infty$, as F is continuous (and in particular left-continuous),

$$F(F^-(t) = \lim_{n \to +\infty} F(x_n) \leq t.$$

Finally, in this case, both $F(F^-(t)) \leq t$ and $F(F^-(t)) \geq t$ imply that $F(F^-(t)) = t$.

v. By applying ii. to $t = F(x)$ leads to $F^-(F(x)) = F^-(t) \leq x$.

Assume know that F is strictly increasing. Assume (absurd) that $F^-(F(x)) < x$. Then

$$F(F^-(F(x))) < F(x),$$

which is impossible by the first point applied to $t = F(x)$.

vi. Immediate from the previous point.

□

B) Back to inhomogeneous Poisson processes

Proposition 4.7.

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a locally integrable function, and denote

$$\Lambda : x \in \mathbb{R}_+ \mapsto \int_0^x \lambda(u)du.$$

Note that Λ is a non-decreasing continuous function.

1. *Let $\tilde{N}$ be a homogeneous Poisson process with (constant) rate 1, and denote*

$$0 < S_1 < S_2 < \ldots < S_n < \ldots$$

its arrival times. Define respectively for all $n \in \mathbb{N}^*$, and for all $t \in \mathbb{R}_+$

$$T_n = \Lambda^-(S_n) \qquad \text{and} \qquad N_t = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{T_n \leq t\}},$$

where Λ^- denote the generalized inverse function of Λ. Then $N = (N_t)_{t \in \mathbb{R}_+}$ is an inhomogeneous Poisson process with intensity function λ.

2. *Conversely, assume that Λ is strictly increasing (and thus a bijective function, since it is always continuous). Let N be a inhomogeneous Poisson process with intensity function λ on $\mathbb{R}_+$. Denote*

$$0 < T_1 < T_2 < \ldots < T_n < \ldots,$$

its arrival times. Define respectively for all n in $\mathbb{N}^*$, and for all t in $\mathbb{R}_+$,

$$S_n = \Lambda(T_n) \qquad \text{and} \qquad \tilde{N}_t = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{S_n \leq t\}}.$$

Then the point process $\tilde{N} = (\tilde{N}_t)_{t \in \mathbb{R}_+}$ is a homogeneous Poisson process with (constant) rate 1.

Proof. In exercise.

1. (a) Let $t > 0$. Express N_t in terms of $(S_n)_{n \in \mathbb{N}^*}$, and deduce, by Property 4.6, that $N_t = \tilde{N}_{\Lambda(t)}$.

Solution.

$$N_t = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{T_n \leq t\}} = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{\Lambda^-(S_n) \leq t\}}.$$

Yet, by point ii. of Property 4.6,

$$\Lambda^-(S_n) \leq t \quad \Leftrightarrow \quad S_n \leq \Lambda(t).$$

Hence

$$N_t = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{S_n \leq \Lambda(t)\}} = \tilde{N}_{\Lambda(t)}.$$

(b) Conclude.

Solution.

i. $N_0 = \tilde{N}_0 = 0$ since $\tilde{N}$ is a homogeneous Poisson process and $\Lambda(0) = \int_0^0 \lambda(u)du = 0$.

ii. Let $t_1 < t_2 \leq s_1 < s_2$. Then, since Λ is non-decreasing, then

$$\Lambda(t_1) \leq \Lambda(t_2) \leq \Lambda(s_1) \leq \Lambda(s_2).$$

In particular, since $\tilde{N}$ is a homogeneous Poisson process,

$$N_{t_2} - N_{t_1} = \tilde{N}_{\Lambda(t_2)} - \tilde{N}_{\Lambda(t_1)} \quad \perp\!\!\!\perp \quad \tilde{N}_{\Lambda(s_2)} - \tilde{N}_{\Lambda(s_1)} = N_{s_2} - N_{s_1}.$$

Note that this remains true if $\Lambda(t_2) = \Lambda(t_1)$ (or if $\Lambda(s_2) = \Lambda(s_1)$) since in this case, the corresponding increment is constant (equal to zero).

iii. Finally, since $\tilde{N}$ is a homogeneous Poisson process with rate 1,

$$N_t = \tilde{N}_{\Lambda(t)} \sim \mathcal{P}\left(1 \times \Lambda(t)\right).$$

2. Assume that Λ is strictly increasing (and thus bijective), and denote Λ^{-1} its inverse function.
   (a) What is the monotony of Λ^{-1}? Deduce that for all u, t,

$$\Lambda(u) \leq t \quad \Leftrightarrow \quad u \leq \Lambda^{-1}(t).$$

Solution.
Since Λ^{-1} is the inverse function of a strictly increasing function, it is also strictly increasing. We deduce that:

$\boxed{\Rightarrow}$ if $\Lambda(u) \leq t$, then $u = \Lambda^{-1}(\Lambda(u)) \leq \Lambda^{-1}(t)$,

$\boxed{\Rightarrow}$ if $u \leq \Lambda^{-1}(t)$, then $\Lambda(u) \leq \Lambda(\Lambda^{-1}(t)) = t$.

   (b) Deduce that for all $t > 0$, $\tilde{N}_t = N_{\Lambda^{-1}(t)}$.

Solution.
Let $t > 0$. Then,

$$\tilde{N}_t = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{S_n \leq t\}} = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{\Lambda(T_n) \leq t\}} = \sum_{n \in \mathbb{N}^*} \mathbb{1}_{\{T_n \leq \Lambda^{-1}(t)\}} = N_{\Lambda^{-1}(t)}.$$

   (c) Conclude.

Solution.

i. Since $\Lambda(0) = 0$, then $\Lambda^{-1}(0) = 0$ and $\tilde{N}_0 = N_0 = 0$.

ii. $\tilde{N}$ has independent increments since N does.

iii. By Proposition 4.2,

$$\tilde{N}_{s+t} - \tilde{N}_s = N_{\Lambda^{-1}(t+s)} - N_{\Lambda^{-1}(s)} \sim \mathcal{P}\left(\int_{\Lambda^{-1}(s)}^{\Lambda^{-1}(s+t)} \lambda(u)du\right),$$

with $\int_{\Lambda^{-1}(s)}^{\Lambda^{-1}(s+t)} \lambda(u)du = \Lambda\left(\Lambda^{-1}(s+t)\right) - \Lambda\left(\Lambda^{-1}(s)\right) = s + t - s = t$. Hence,

$$\tilde{N}_{s+t} - \tilde{N}_s \sim \mathcal{P}(1 \times t).$$

We deduce from all three points above that $\tilde{N}$ is a homogeneous Poisson process with (constant) rate 1.

$\square$

II.2 Acceptance/Rejection (or Thinning)

The following result leads to one of the most popular method for generating an inhomogeneous Poisson process, sometimes called the *thinning method*. It is the "point process" analogue of the acceptance/rejection method classically used to simulate r.v.

It generalizes Proposition 2.27 which claims that if you randomly classify each point of a homogeneous Poisson process with rate λ into two categories (with probability p and $1-p$), the two corresponding point processes are also homogeneous Poisson processes with corresponding rates $p\lambda$ and $(1-p)\lambda$. In the inhomogeneous case, we allow p to be a varying function of time.

Theorem 4.8.

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a locally integrable non-negative function. Assume that λ is upper-bounded by $M \in \mathbb{R}_+^*$ on a subset $[0,T)$ of $\mathbb{R}_+$ ($T \in \mathbb{R}_+^* \cup \{+\infty\}$), that is

$$\forall u \in [0,T), \quad \lambda(u) \leq M.$$

Let N be a homogeneous Poisson process with rate M and classify each point u of N as type I *with probability $p(u) := \lambda(u)/M$ and type* II *otherwise. Then, the resulting point processes N^{I} and N^{II} of type* I *and type* II *events satisfy*

a) N^{I} is a inhomogeneous Poisson process with intensity function λ.
N^{II} is a inhomogeneous Poisson process with intensity function $M-\lambda$.

b) $N^{\mathrm{I}} \perp\!\!\!\perp N^{\mathrm{II}}$.

Idea of proof. a) The independence of the increments comes from the independence of the increments of the homogeneous Poisson process N, and the independence between the tags.

b) Let us now prove the Poisson distribution, and that $N^{\mathrm{I}} \perp\!\!\!\perp N^{\mathrm{II}}$.

- **Modeling:** We know (c.f. Proposition 2.23) that conditionally on $\{N_t = n\}$, $T_1, \ldots, T_n$ behaves as the order statistic $U_{(1)} \leq \ldots \leq U_{(n)}$ associated to n i.i.d. r.v. $U_1, \ldots, U_n$ with distribution $\mathcal{U}([0,T])$. The classifying process can be seen as follows: Let $Y_1, \ldots, Y_n$ be n i.i.d. r.v. with distribution $\mathcal{U}([0,M])$, independent on $(U_1, \ldots, U_n)$.

$$\text{Classify } U_i \text{ as} \quad \begin{cases} \text{type I if } Y_i \leq \lambda(U_i), \\ \text{type II otherwise.} \end{cases}$$

Then,

$$\begin{aligned}
\mathbb{P}(U_i \text{ of type I} | U_i = u) &= \mathbb{P}(Y_i \leq \lambda(U_i) | U_i = u) \\
&= \mathbb{P}(Y_i \leq \lambda(u) | U_i = u) \\
&= \mathbb{P}(Y_i \leq \lambda(u)), && \text{as } Y_i \perp\!\!\!\perp U_i \\
&= \frac{\lambda(u)}{M} = p(u) && \text{as } Y_i \sim \mathcal{U}([0,M]).
\end{aligned}$$

- **Towards a binomial distribution:** Conditionally on $\{N_t = n\}$, we thus obtain

$$N_t^{\mathrm{I}} = \sum_{i=1}^{n} \mathbb{1}_{\{Y_i \leq \lambda(U_i)\}},$$

where the $\mathbb{1}_{\{Y_i \le \lambda(U_i)\}}$ are independent (by independence of the couples $(U_i, Y_i)_i$) Bernoulli r.v. with parameter

$$\tilde{p} := \quad \mathbb{P}(Y_i \le \lambda(U_i)) = \int_0^{+\infty} \underbrace{\mathbb{P}(Y_i \le \lambda(U_i)|U_i = u)}_{p(u)} \underbrace{f_{U_i}(u)}_{\frac{1}{t}\mathbb{1}_{\{u\in[0,t]\}}} du = \frac{1}{t}\int_0^t p(u)du.$$

We deduce that the conditional distribution of N_t^{I} given $\{N_t = n\}$, is a binomial distribution with parameters n and $\tilde{p}$, that is

$$N_t^{\mathrm{I}} \sim \mathcal{B}(n, \tilde{p}).$$

- **Unconditioning:** Let $k, l \in \mathbb{N}$. Then

$$\begin{aligned}
\mathbb{P}\big(N_t^{\mathrm{I}} = k, N_t^{\mathrm{II}} = l\big) &= \mathbb{P}\big(N_t^{\mathrm{I}} = k, N_t = k + l\big) \\
&= \mathbb{P}\big(N_t^{\mathrm{I}} = k \big| N_t = k + l\big)\, \mathbb{P}(N_t = k + l) \\
&= \binom{k+l}{k} \tilde{p}^k (1 - \tilde{p})^l \times \frac{(Mt)^{k+l}}{(k+l)!} e^{-Mt(\tilde{p}+1-\tilde{p})} \\
&= \left[\frac{(Mt\tilde{p})^k}{k!} e^{-Mt\tilde{p}}\right] \times \left[\frac{(Mt[1-\tilde{p}])^l}{l!} e^{-Mt(1-\tilde{p})}\right].
\end{aligned}$$

We deduce that

$$\mathbb{P}\big(N_t^{\mathrm{I}} = k\big) = \sum_{l=0}^{+\infty} \mathbb{P}\big(N_t^{\mathrm{I}} = k, N_t^{\mathrm{II}} = l\big) = \frac{(Mt\tilde{p})^k}{k!} e^{-Mt\tilde{p}},$$

thus

$$N_t^{\mathrm{I}} \sim \mathcal{P}(Mt\tilde{p}), \qquad \text{with} \qquad Mt\tilde{p} = Mt\frac{1}{t}\int_0^t \frac{\lambda(u)}{M} du = \int_0^t \lambda(u)du.$$

In the same way, we prove that $N_t^{\mathrm{II}} \sim \mathcal{P}(Mt[1 - \tilde{p}])$, with

$$Mt[1 - \tilde{p}] = Mt - \int_0^t \lambda(u)du = \int_0^t [M - \lambda(u)]\, du.$$

Finally, we also deduce that $N_t^{\mathrm{I}} \perp\!\!\!\perp N_t^{\mathrm{II}}$ because of that product form above.

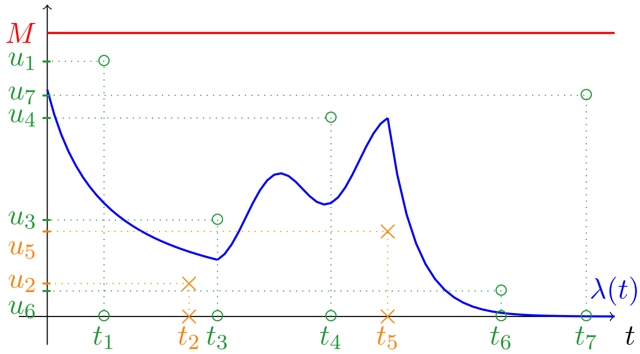
□

Algorithm 1 *Thinning algorithm*

Input: intensity function λ; simulation interval $[0, T]$; upper-bound $M > 0$.

1. Generate a homogeneous Poisson process with constant rate M on $[0, T]$;
   Denote $t_1, \ldots, t_n$ the observed arrival times on $[0, T]$.
2. Generate independently n i.i.d r.v. with distribution $\mathcal{U}[0, M]$;
   Denote $u_1, \ldots, u_n$ the observed values.
3. for i in 1 to n, keep t_i if $u_i \le \lambda(t_i)$.
4. Order the kept points.

Output: the set of ordered kept points are the arrival times of an inhomogeneous Poisson process with intensity function λ on $[0, T]$.



Observed arrival times:

- N: $t_1, \dots, t_7$;
- N^{I}: t_2, t_5;
- N^{II}: t_1, t_3, t_4, t_6, t_7.

III Arrival times

III.1 First arrival time after a given instant

Property 4.9.

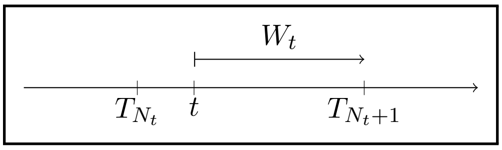
Let N be a Poisson process with intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$.

i. Then the survival function of the first arrival time T_1 equals for all $s > 0$,

$$\mathbb{P}(T_1 > s) = \mathbb{P}(N_s = 0) = e^{-\int_0^s \lambda(u)du} = e^{-\Lambda(s)}.$$

It is not exponentially distributed anymore (except if $\Lambda(s) = \lambda s$, i.e. if the Poisson process is homogeneous). Note that λ is the hazard rate of T_1.

ii. Denote W_t the waiting time at time t, that is



$$W_t = T_{N_t+1} - t.$$

Then, for all $s > 0$,

$$\mathbb{P}(W_t > s) = \mathbb{P}(N_{t+s} - N_t = 0) = e^{-\int_t^{s+t} \lambda(u)du} = e^{-\int_0^s \lambda(u+t)du}.$$

In particular, from Proposition 1.14, the hazard rate of W_t is

$$h_t : u \mapsto \lambda(u + t).$$

III.2 (Conditional) distribution of the arrival times

Proposition 4.10.

Let N be an inhomogeneous Poisson process with intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$. Fix $n \in \mathbb{N}^*$ and denote $(T_1, \dots, T_n)$ the n first arrival times. Then $(T_1, \dots, T_n)$ has the following joint density w.r.t. the Lebesgue mesure on $\mathbb{R}^n$:

$$(t_1, \dots, t_n) \mapsto e^{-\displaystyle\int_0^t \lambda(u)du} \left(\prod_{i=1}^n \lambda(t_i) \right) \mathbb{1}_{\{0<t_1<\dots<t_n\}}.$$

Note that if λ is a constant function, we recover the density in the continuous case (c.f. Lemma 2.19).

Proof. Admitted here.
Main argument: mathematical induction on n, using the Time change Proposition 4.7. □

We can thus deduce the conditional distribution of the arrival times.

Proposition 4.11.

Let $n \in \mathbb{N}^*$ and $t \in \mathbb{R}_+^*$. Given that $\{N_t = n\}$, the n arrival times $0 < T_1 < \ldots < T_n$ in $[0, t]$ of an inhomogeneous Poisson process with intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, has conditional density w.r.t. the Lebesgue measure on $\mathbb{R}^n$:

$$(t_1, \ldots, t_n) \mapsto \frac{n!}{\int_0^t \lambda(u)du} \left(\prod_{i=1}^{n} \lambda(t_i) \right) \mathbb{1}_{\{0<t_1<\ldots<t_n\leq t\}}.$$

In particular, conditionally on $\{N_t = n\}$, $(T_1, \ldots, T_n)$ behaves as the order statistic associated to n i.i.d. r.v. with common density

$$s \in \mathbb{R} \mapsto \frac{\lambda(s)}{\int_0^t \lambda(u)du} \mathbb{1}_{\{0<s\leq t\}}.$$

Proof. Similar to the one of Proposition 2.23. □

This result leads to a new construction of an inhomogeneous Poisson process.

Proposition 4.12.

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be an intensity function. Let $Y \sim \mathcal{P}\left(\int_0^T \lambda(u)du\right)$.

- *If $Y = 0$, then for all $0 \leq t \leq T$, $N_t = 0$ (the point process has no points in $[0, T]$.*
- *Otherwise, let $S_1, \ldots, S_Y$ be i.i.d. r.v. with density w.r.t. the Lebesgue measure on $\mathbb{R}$*

$$s \in \mathbb{R} \mapsto \frac{\lambda(s)}{\int_0^T \lambda(u)du} \mathbb{1}_{\{0<s\leq T\}}.$$

Sort the values $(T_1, \ldots, T_Y) = (S_{(1)}, \ldots, S_{(Y)})$ and consider the associated counting process defined for all $0 \leq t \leq T$,

$$N_t = \sum_{n=1}^{Y} \mathbb{1}_{\{T_n\leq t\}} = \sum_{i=1}^{Y} \mathbb{1}_{\{S_i\leq t\}}.$$

Then $(N_t)_{t\in[0,T]}$ is an inhomogeneous Poisson process with intensity function λ on $[0, T]$.

Proof.

- One the one hand, on the event $\{Y = 0\}$, $N_t = 0$.
- On the other hand, on the event $\{Y = n\}$,

$$N_t = \sum_{i=1}^{n} \mathbb{1}_{\{S_i\leq t\}}.$$

To prove that $(N_t)_{t\in[0,T]}$ is a Poisson process, we need to prove that for all subdivision of $[0, T]$,

$$0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = T,$$

and for all integers $k_1, \dots, k_m$, one has

$$\mathbb{P}\big(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m\big) = \prod_{j=1}^{m} \left(\frac{c_j^{k_j}}{k_j!} e^{-c_j} \right), \qquad (4.1)$$

where

$$c_j = \int_{t_{j-1}}^{t_j} \lambda(u)du = \Lambda(t_j) - \Lambda(t_{j-1}).$$

Yet, since $N_{t_m} = N_T = Y$ (there are Y S_i's that belong to $[0,T]$),

$$\begin{aligned}\mathbb{P}\big(N_{t_1} = k_1, &N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m\big) \\ &= \mathbb{P}\left(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m, Y = \sum_{j=1}^{m} k_j \right) \\ &= \mathbb{P}\left(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m \middle| Y = \sum_{j=1}^{m} k_j \right) \mathbb{P}\left(Y = \sum_{j=1}^{m} k_j \right)\end{aligned}$$

On the one hand, for all j, for all i, by definition of the S_i's,

$$\mathbb{P}(S_i \in (t_{j-1}, t_j]) = \int_{t_{j-1}}^{t_j} \frac{\lambda(s)}{\Lambda(T)} ds = \frac{c_j}{c},$$

where c denotes

$$c = \int_0^T \lambda(u)du = c_1 + \dots + c_m = \Lambda(T).$$

Hence, conditionally on $\left\{ Y = \sum_{j=1}^{m} k_j \right\}$, the random variables $\left(\sum_{i=1}^{k_1+\dots+k_m} \mathbb{1}_{\{S_i \in (t_{j-1}, t_j]\}} \right)_{1 \le j \le m}$, which count the number of S_i's in each interval $(t_{j-1}, t_j]$, have a multinomial distribution with parameters $(k_1 + \dots + k_m)$ and $(c_j/c)_{1 \le j \le m}$. Therefore,

$$\begin{aligned}\mathbb{P}\left(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m \middle| Y = \sum_{j=1}^{m} k_j \right) \\ = \mathbb{P}\left(\sum_{i=1}^{k_1+\dots+k_m} \mathbb{1}_{\{S_i \in (t_0, t_1]\}} = k_1, \dots, \sum_{i=1}^{k_1+\dots+k_m} \mathbb{1}_{\{S_i \in (t_{m-1}, t_m]\}} = k_m \right) \\ = \frac{(k_1 + k_2 + \dots + k_m)!}{k_1! \dots k_m!} \left(\frac{c_1}{c} \right)^{k_1} \dots \left(\frac{c_m}{c} \right)^{k_m}\end{aligned}$$

On the other hand, $Y \sim \mathcal{P}(\Lambda(T))$, with $\Lambda(T) = c = c_1 + \dots + c_m$.

Finally,

$$\begin{aligned}\mathbb{P}\big(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2, \dots, N_{t_m} - N_{t_{m-1}} = k_m\big) \\ = \frac{(k_1 + k_2 + \dots + k_m)!}{k_1! \dots k_m!} \left(\frac{c_1}{c} \right)^{k_1} \dots \left(\frac{c_m}{c} \right)^{k_m} \frac{c^{\sum_{j=1}^{m} k_j}}{(k_1 + \dots + k_m)!} e^{-c} \\ = \frac{c_1^{k_1}}{k_1!} e^{-c_1} \times \dots \times \frac{c_m^{k_m}}{k_m!} e^{-c_m},\end{aligned}$$

which ends the proof of (4.1). □

IV Statistics for inhomogeneous Poisson processes

As in the homogeneous case, there are two types of observation: either n is fixed (and the observation window is random), or the observation window is fixed (and you observe a random number of events). Let us focus on the fixed window case.

If one has a parametric model $\{\lambda_\theta; \theta \in \Theta\}$, one can introduce the Maximum Likelihood Estimator (MLE). To do so, we need to compute the Likelihood of the unknown parameter θ.

Proposition 4.13.
Let N be an inhomogeneous Poisson process with intensity function $\lambda_\theta : \mathbb{R}_+ \to \mathbb{R}_+$ depending on an unknown parameter θ in Θ. Consider the number N_T of "events" that have occurred in $[0, T]$, dans denote $(T_1, \ldots, T_{N_T})$ the corresponding arrival times. Then likelihood of the trajectory of N is equal to

$$\mathcal{L}\left((N_t)_{t\in[0,T]}; \theta\right) = \left(\prod_{n=1}^{N_T} \lambda_\theta(T_n)\right) e^{-\int_0^T \lambda_\theta(u)du}.$$

Then, if it exists, one can compute the MLE of θ,

$$\hat{\theta}_{MLE} \in \operatorname{argmax}_{\theta\in\Theta} \mathcal{L}\left((N_t)_{t\in[0,T]}; \theta\right),$$

and estimate the intensity function by $\lambda_{\hat{\theta}_{MLE}}$.

You will have the opportunity to deepen your understanding of statistical inference for inhomogeneous Poisson processes through various mini-projects.