INSA, 5gmm.

Lifetime Data Analysis

Problem Set 2: Exponential distribution, Estimation

Exercise 1

An electronic equipment has a lifetime T with exponential distribution. The parameter λ is estimated by engeniers at $\lambda = 0.25$ per year.

1. Give the M.T.T.F. (Mean Time To Failure, defined as $\mathbb{E}T$) and plot (e.g. with Excel) the Reliability function defined by $t \mapsto R(t) = P(T > t)$. Check that you understand how to use this curve.
2. Give the probability to reach the M.T.T.F. without failure.
3. What is the optimal time (in months) of renewal to get less than 5% of returns due to failure?
4. We now suppose that the lifetime distribution is Weibull with parameters $\alpha = 4$ and $\beta = 1.1$. Give the M.T.T.F.[1] Plot the Reliability function in this case.
5. Find the new optimal time of renewal (under the same constraint).

Exercise 2

Assume that items are produced at two different plants. All the items are supposed to have independent behavior and a lifetime with exponential distribution. However, due to differences in the production process, the parameter of the exponential distribution is $\lambda_1 = 0.1$ per day for the plant 1 and $\lambda_1 = 0.5$ for plant 2. We select randomly an item, knowing that $p = 30\%$ of the whole production come from plant 1.

1. Write the reliability of the item as a function of λ_1, λ_2 and p.
2. Plot the Reliability functions of Plant 1, Plant 2 and of a randomly chosen item from one of the two plants. Plot also the overall conditional reliability at age $t = 3$. What do you observe?
3. Find the M.T.T.F. of the item selected randomly.

Exercise 3

Give the Maximum Likelihood Estimator of the parameter λ of an exponential distribution in the two following cases. In the first case, we suppose that a complete sample $X_1, \ldots, X_n$ is observed. In the second case, this is a randomly right censored sample $(T_1, \delta_1), \ldots, (T_n, \delta_n)$ with the assumption that the censoring distribution doesn't depend on λ. Do you catch the intuitive interpretation of these estimators? What happens now if the observations are not right censored but left censored by deterministic times?

[1] You can use the lngamma function of EXCEL which gives the logarithm of $\Gamma(x)$

Exercise 4

A life test has been carried out on a lot of 11 lamps. These lamps are assumed to be identical (same characteristics, manufactured by the same company etc...) and to behave independently. The observed lifetimes (in hours) are given in Table 1:

0.5	2.75	1.3	11.05	3.5	5.25	1.75	6.25	8.5	15	20.5

Table 1: Observed lifetimes of 11 lamps on test.

1. Estimate the M.T.T.F. Give an asymptotic 95% confidence interval for the M.T.T.F.
2. Remind what is $R(10)$. Give an estimate of it. Plot (e.g. with Excel) the estimate of the function $R(t)$ with respect of time t.
3. Fix t and write $N(t)$ the number of x_i's strictly greater than t.
   (a) What is the distribution of $N(t)$?
   (b) How can you justify that for large n, the random variable $N(t)/n$ has approximatively a normal distribution? Specify the parameters of this normal distribution. Deduce from that a confidence interval for $R(t)$. Specify this interval for $t = 10$.
   (c) Plot these confidence intervals with respect to t in the previous figure.
4. Now, suppose that the lifetime has an exponential distribution with parameter λ. Give an estimate of λ.
5. We still assume an exponential distribution on the lifetime.
   (a) Give another estimate of $R(10)$. Add on the previous plot the new estimation of the Reliability function $R(t)$ you get assuming an exponential distribution.
   (b) We can show that $2n\lambda\bar{X}$ has a $\chi^2(2n)$ distribution. Give a 95% confidence interval for λ. Deduce from that a new 95% confidence interval for the M.T.T.F.
   (c) Finally, deduce a 95% confidence interval for $R(t)$. Specify this interval for $t = 10$. On a same figure, plot the different estimations and confidence intervals you obtained from the beginning.
6. *Redo*[2] *all of that with a sample of size 100 of simulated data.*
   (a) First consider an exponential distribution with parameter $\lambda = 0.25$.
   (b) Use now a Weibull distribution with parameters $\alpha = 4$ *and* $\beta = 2$.

Exercise 5

A 2 hours life test has been carried out on n repairable items. We assume that the lifetimes of the items are exponentially distributed with unknown but same parameter λ. When a failure occurred on an item, it is instantaneously replaced by a new one. Write $k_1, \ldots, k_n$ the number of failures observed on each item during the test.

1. Recall the distribution of the number of failures observed on an item during the first 2 hours.
2. Give the expression of maximum likelihood estimator of λ as function of the observations $k_1, \ldots, k_n$.
3. Suppose that the observed empirical mean $\bar{k}$ is 1.986. Give the value of the estimate of λ.

[2] Falcutatif