Chapitre 3 : Stability Analysis

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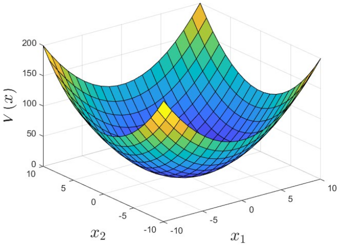

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version : 10-2022

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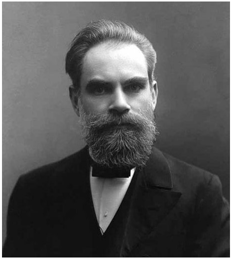
Introduction

- Stability is an essential concept in automatic control theory
  - $\hookrightarrow$ for instance, first requirement in closed-loop control
- It exists several notions of stability
  - $\hookrightarrow$ stability of an equilibrium point / input-output stability
- Main method : **Lyapunov theory**
  - $\hookrightarrow$ A.M. Lyapunov (1857-1918) is Russian mathematician

defended his PhD thesis in 1885
at the University of St Petersbourg
under supervision of P. Tchebychev



Introduction

We still consider autonomous systems, without input

$$\dot{x} = f(x) \qquad \text{with initial conditions : } x(0) = x_0$$

where it is assumed that

- f is locally Lipschitz in a domain $\mathcal{D} \subset \mathbb{R}^n$
- x^* is an equilibrium point, that is $f(x^*) = 0$

Without loss of generality, we will consider in the sequel that

$$x^* = 0$$

In deed, if $x^* \neq 0$, by change of variable $y = x - x^*$

$$\dot{y} = \dot{x} = f(y + x^*) \stackrel{def}{=} g(y) \quad \text{where } g(0) = 0$$

Definitions

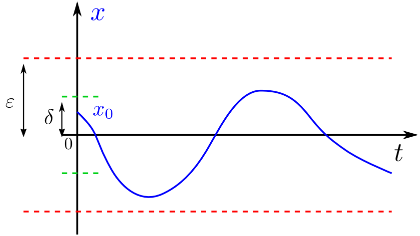
Behavior of trajectories of x around the equilibrium point ?

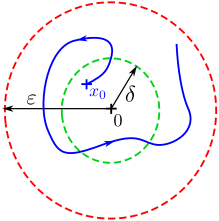
Stability

The equilibrium point 0 is said **stable** if

$$\forall \epsilon > 0, \quad \exists \delta = \delta(\epsilon) > 0 \ \text{ such that } \ \|x(0)\| < \delta \ \Rightarrow \ \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

Solutions remain bounded if the initial condition is small enough





Definitions

What about convergence to the equilibrium point ?

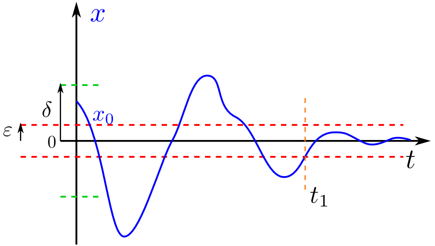
Attractivity

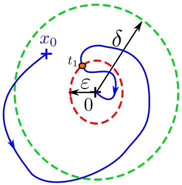
The equilibrium point 0 is said to be **attractor** if

$$\exists \delta > 0, \quad \|x(0)\| < \delta \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0$$

$$\text{or} \quad \exists \delta > 0, \quad \|x(0)\| < \delta \;\Rightarrow\; \forall \epsilon > 0, \quad \exists t_1 > 0 \;\text{ such that }\; \forall t > t_1, \quad \|x(t)\| < \epsilon$$

Solutions converge to 0 for $t \to \infty$ if the initial condition is small enough





Definitions

Asymptotic stability

The equilibrium point 0 is said to be **asymptotically stable** if it is stable and attractor

Unstability

The equilibrium point 0 is said **unstable** if it is not stable

- Stability is a notion that is **local**
- Attractivity is a notion that can be **local** or **global**
- If from any initial conditions $x_0 \in \mathbb{R}^n$ the equi. pt is attractor, then it is said **globally asymptotically stable** (GAS). It is LAS otherwise.
- The set of initial conditions such that the equilibrium point is AS is called the **region of attraction**

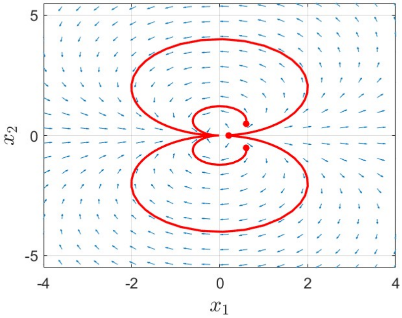
Stability and attractivity

Stability and attractivity are two different notions

- stability looks at whether the trajectories remain in some neighbourhood of the equilibrium
- attractivity looks at whether the trajectories converge to the equilibrium

Butterfly system : unique equilibrium point 0 is globally attractor but unstable

$$\begin{cases} \dot{x}_1 = x_1^2 - x_2^2 \\ \dot{x}_2 = 2x_1x_2 \end{cases}$$

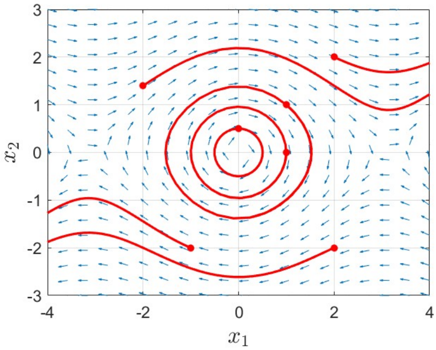


Stability and attractivity

Consider system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}$$

- Equilibrium points : $x^\star = [k\pi \ , \ 0]^T$, $k \in \mathbb{Z}$
- Equilibrium point is **stable but not attractor**



Another definition

Exponential stability

The equilibrium point 0 is said to be **exponentially stable** if it exists two strictly positive scalars α and k such that

$$\exists \delta > 0, \quad \|x(0)\| < \delta \quad \Rightarrow \quad \forall t \geq 0, \quad \|x(t)\| < k\|x(0)\|e^{-\alpha t}$$

Consider system :

$$\dot{x} = -\left(1 + \sin^2(t)\right)x$$

Solution : $x(t) = x(0)e^{-\int_0^t 1+\sin^2(\tau)d\tau}$

$\Rightarrow$ **Exponential stability** :

$$\|x(t)\| < \|x(0)\|e^{-t} \qquad \text{since} \qquad \int_0^t 1 + \sin^2(\tau)d\tau > t$$

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First remarks

- The objective is to study the convergence of the system trajectories towards the origin (equilibrium point of interest) without explicit description of these trajectories.

  $\hookrightarrow$ no need to solve any differential equation

- For linear systems, stability can be assessed with eigenvalues of the dynamic matrix A. Could we use linear approximation to analyze the stability (at least local) of a nonlinear system ?

  $\hookrightarrow$ the **first method of Lyapunov** can answer this question

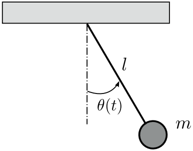
- For nonlinear systems, a more general result is actually required

  $\hookrightarrow$ the **second method of Lyapunov** is a powerful tool

Introductory example : pendulum

State variables : $x_1 = \theta$ and $x_2 = \dot{\theta}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$



Let us calculate the energy of the system $E(x)$ = potential energy + kinetic energy

$$\begin{aligned} E(x) &= mgl(1-\cos\theta) + \frac{1}{2}ml^2\dot{\theta}^2 \\ &= mgl(1-\cos x_1) + \frac{1}{2}ml^2x_2^2 \end{aligned}$$

How it evolves in time ?

$$\frac{dE(x)}{dt} = \frac{dE(x)}{dx}\frac{dx}{dt} = \begin{bmatrix} mgl\sin x_1 & ml^2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix} = -kl^2x_2^2$$

◇ What can we conclude ?

Introductory example : pendulum

$$E(x) = mgl(1 - \cos x_1) + \frac{1}{2} ml^2 x_2^2 \quad (> 0)$$

$$\frac{dE(x)}{dt} = -kl^2 x_2^2 \quad (\leq 0)$$

- The energy derivative is negative or zero $\Rightarrow$ trajectories won't diverge
- if $k = 0$, $\frac{dE}{dt} = 0$ along system trajectories $\Rightarrow$ conservation of mechanical energy

  $\hookrightarrow$ equilibrium point 0 is stable
- if $k > 0$, $\frac{dE}{dt} \leq 0$ along system trajectories $\Rightarrow$ energy is decreasing until $E = 0$

  $\hookrightarrow$ equilibrium point 0 is asymptotically stable

◇ Extension to more general functions (than energy functions) : **Lyapunov functions**

Fundamental theorem of stability (local)

Theorem

Consider an equilibrium point $x^* = 0$ and a domain $\mathcal{D} \subset \mathbb{R}^n$ including 0.
Let $V : \mathcal{D} \to \mathbb{R}$, be a C^1 function such that :

$$V(x^*) = 0 \qquad \text{and} \qquad V(x) > 0 \quad \forall x \in \mathcal{D} \setminus \{0\}$$

$$\dot{V}(x) \leq 0 \quad \forall x \in \mathcal{D}$$

then x^* is a **stable** equilibrium point. Moreover, if

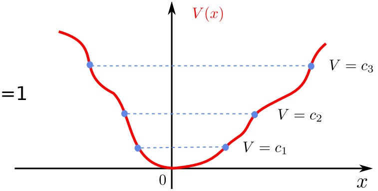
$$\dot{V}(x) < 0 \quad \forall x \in \mathcal{D} \setminus \{0\}$$

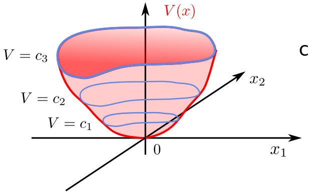
then x^* is **asymptotically stable**.

- We consider here **local stability** (domain $\mathcal{D}$)
- A function V satisfying the above conditions is a Lyapunov function
- This result provides only a **sufficient condition** for stability !

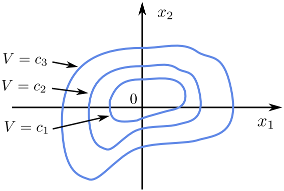
Illustration of the shape of a Lyapunov function

cas n=1





cas n=2

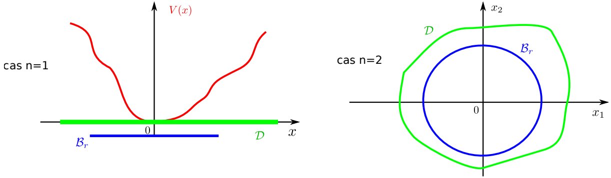


- State domain definition : $x \in \mathcal{D}$
- Lyapunov function : $V(x) \geq 0$, but $= 0$ only at x^*, and $\dot{V}(x) \leq 0$ (or < 0)
- $V(x) = c_i$ are level curves

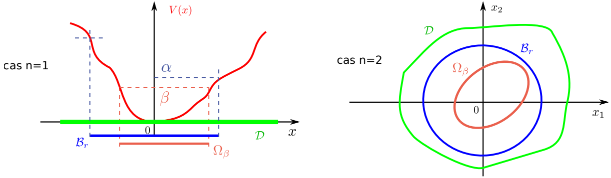
Sketch of the proof (1)

Having in mind the definition of the stability

- Given a $\varepsilon > 0$ and define $r \in (0, \varepsilon)$ such that $\mathcal{B}_r \subset \mathcal{D}$ with $\mathcal{B}_r = \{x \in \mathbb{R}^n, \ \|x\| \leq r\}$



- Let be $\alpha = \min_{\|x\|=r} V(x)$ (> 0) and define $\beta \in (0, \alpha)$ with $\Omega_\beta = \{x \in \mathcal{B}_r, \ V(x) \leq \beta\}$

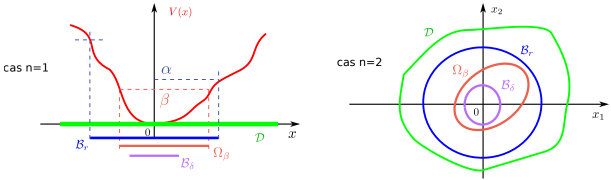


- By definition $\Omega_\beta \subset \mathcal{B}_r$ and all trajectories initiated in Ω_β remain in Ω_β since

$$\dot{V}(x) \leq 0 \quad \Rightarrow \quad V(x(t)) \leq V(x(0)) \leq \beta \qquad \forall t \geq 0$$

Sketch of the proof (2)

- It exists $\delta > 0$ such that $\|x\| \leq \delta \Rightarrow V(x) \leq \beta$ (set $\mathcal{B}_\delta$)



- Then $\mathcal{B}_\delta \subset \Omega_\beta \subset \mathcal{B}_r$ and

$$x(0) \in \mathcal{B}_\delta \quad \Rightarrow \quad x(0) \in \Omega_\beta \quad \Rightarrow \quad x(t) \in \Omega_\beta \quad \Rightarrow \quad x(t) \in \mathcal{B}_r$$

- Hence, we can conclude that the equilibrium point is stable since

$$\|x(0)\| \leq \delta \quad \Rightarrow \quad \|x(t)\| \leq r < \varepsilon \qquad \forall t \geq 0$$

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Some vocabulary

- A function s.t. $V(0) = 0$ and $V(x) > 0 \quad \forall x \neq 0$ is a *positive definite function*
- A function s.t. $V(0) = 0$ and $V(x) \geq 0 \quad \forall x \neq 0$ is a *positive semi-definite funct.*
- A function s.t. $V(0) = 0$ and $V(x) < 0 \quad \forall x \neq 0$ is a *negative definite funct.*
- A function s.t. $V(0) = 0$ and $V(x) \leq 0 \quad \forall x \neq 0$ is a *negative semi-definite funct.*
- Note that $V(x)$ negative semi-definite $\equiv -V(x)$ positive semi-definite
- The surface $V(x) = c$ is called a level line (or surface) of the function

Examples :

- $V(x) = (x_1 + x_2)^2$ is positive semi-definite in $\mathbb{R}^2$
- $V(x) = x_1^2 + x_2^2$ is positive definite in $\mathbb{R}^2$
- $V(x) = x_1^2 + x_2^2 - 4$ is negative definite in any disc (in $\mathbb{R}^2$) of radius < 2

An hermitian matrix P is positive definite (semi-definite) iff

$$x^T P x > 0 \quad (\geq 0), \qquad \forall x \neq 0$$

Note also that $P > 0$ iff all its leading principal minors are strictly positive

Some properties :

- $P > 0 \Leftrightarrow$ all its eigenvalues are real and positive, $\lambda(P) \in \mathbb{R}_*^+$
- $P \geq 0 \Leftrightarrow$ some ev. are 0, others are real and positive, $0 \in \lambda(P)$, $\lambda(P) \in \mathbb{R}^+$
- $P > 0 \Leftrightarrow -P$ is negative definite, $-P < 0$
- $P > 0 \Leftrightarrow P^{-1}$ is positive definite, $P^{-1} > 0$
- $P \geq 0 \Rightarrow P$ is a singular matrix, $\det(P) = 0$
- $M \in \mathbb{C}^{n \times n}$ and $\det(M) \neq 0 \Rightarrow P = M^T M > 0$ is positive definite
- $M \in \mathbb{C}^{n \times n}$ and $\det(M) = 0 \Rightarrow P = M^T M \geq 0$ is positive semi-definite

Example 1

Consider the system (simple linear scalar system)

$$\dot{x} = ax, \qquad a < 0$$

- Let's propose the Lyapunov function candidate : $V(x) = \frac{1}{2}x^2$

  (obviously $V(x) > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$ and $V(0) = 0$)

- Its time-derivative along the trajectories of the system is

$$\dot{V}(x) = x\dot{x} = ax^2$$

- Since $a < 0 \Rightarrow \dot{V}(x) < 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

  $\hookrightarrow$ the above system is asymptotically stable

Example 2

Back to the inverted pendulum

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a \sin x_1 - b x_2 \end{bmatrix}, \qquad \text{with } a \text{ and } b > 0$$

Consider the Lyapunov function candidate : $V(x) = a\big(1 - \cos(x_1)\big) + \frac{1}{2} x_2^2$

- Determine $\mathcal{D}$
- Is V a Lyapunov function for our system ?
- What about this second function :

$$V(x) = a\big(1 - \cos(x_1)\big) + \frac{1}{2} x^T P x, \qquad \text{with } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

P is a positive definite matrix : $p_{11} > 0$, $p_{22} > 0$ and $p_{11} p_{22} - p_{12}^2 > 0$

- Determine $\mathcal{D}$
  - $V(0) = 0$ for $x_1 = 0 \pm 2k\pi$ and $x_2 = 0$
  - $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$ and for any x_2

- Is V a Lyapunov function ?
  - Time derivative :

$$\begin{aligned}\dot{V}(x) &= a\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= -\,bx_2^2\end{aligned}$$

  - Then $\dot{V}(x) \leq 0$, $\dot{V}(0)$ for $x_2 = 0$ and $\forall x_1$
  - Hence, V is a Lyapunov function and the origin of the system is stable

- What about this second function : $V(x) = a\big(1 - \cos(x_1)\big) + \frac{1}{2}x^T P x$

  - Since P is positive definite matrix, $x^T P x > 0\ \forall x \neq 0$
  - For $x \in \mathcal{D}$, $V(0) = 0 \Rightarrow x_1 = x_2 = 0$
  - Time derivative :

$$\begin{aligned}
\dot{V}(x) &= a\sin(x_1)\dot{x}_1 + \frac{1}{2}(\dot{x}^T P x + x^T P \dot{x}) \\
&= a\sin(x_1)\dot{x}_1 + (x_1 p_{11} + x_2 p_{12})\dot{x}_1 + (x_1 p_{12} + x_2 p_{22})\dot{x}_2 \\
&= a\sin(x_1)x_2 + (x_1 p_{11} + x_2 p_{12})x_2 + (x_1 p_{12} + x_2 p_{22})\big(-a\sin(x_1) - bx_2\big) \\
&= a\sin(x_1)x_2(1 - p_{22}) + x_1x_2(p_{11} - bp_{12}) + x_2^2(p_{12} - bp_{22}) - a\sin(x_1)x_1p_{12}
\end{aligned}$$

  For a specific choice of P :

$$\begin{cases} 1 - p_{22} = 0 \\ p_{11} - bp_{12} = 0 \\ p_{12} - bp_{22} < 0 \ \rightarrow \ p_{12} = \frac{b}{2} \end{cases} \quad \Rightarrow \quad P = \begin{bmatrix} \frac{b^2}{2} & \frac{b}{2} \\ \frac{b}{2} & 1 \end{bmatrix}$$

  we have

$$\dot{V}(x) = -\frac{b}{2}x_2^2 - \frac{ab}{2}\sin(x_1)x_1 < 0$$

  - The origin is asymptotically stable

Global asymptotic stability

Previous theorems considered **local** stability (for a region $\mathcal{D}$)

$\hookrightarrow$ What are the conditions to have a **global** property ($\mathcal{D} = \mathbb{R}^n$) ?

Theorem

Let us consider the equilibrium point $x^* = 0$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that

$$\|x(t)\| \to +\infty \quad \Rightarrow \quad V(x) \to +\infty$$

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \neq 0$$

$$\dot{V}(x) < 0 \quad \forall x \neq 0$$

then the origin is **globally asymptotically stable** (GAS).

This fist condition means that function V is *radially unbounded*

Exercise 1

Consider system :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1(x_1^2 + x_2^2) \\ -x_1 - x_2(x_1^2 + x_2^2) \end{bmatrix}$$

- Considering equilibrium point $(0\,,\,0)$, show that

  $$V(x) = x_1^2 + x_2^2$$

  is a Lyapunov function

- Is the stability property asymptotic or not ? local or global ?

Solution :

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Question

Back again on the pendulum example :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a \sin x_1 - b x_2 \end{bmatrix}, \qquad \text{with } a \text{ and } b > 0$$

with the Lyapunov function : $V(x) = a\big(1 - \cos(x_1)\big) + \frac{1}{2}x_2^2$

It was shown that the origin is stable : $\dot{V}(x) = -bx_2^2 \leq 0$

◇ Can we show that the equilibrium point is actually asymptotically stable with the same Lyapunov function ?

LaSalle invariance principle

Definition (invariant set)

A set M is said to be **invariant** if

$$x(0) \in \mathcal{M} \quad \Rightarrow \quad x(t) \in \mathcal{M} \quad \forall t$$

Theorem

Assume there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{V}(x) \leq W(x) \leq 0 \qquad \forall x \in \mathbb{R}^n$$

then

- $x^* = 0$ is a stable equilibrium point
- the solutions of the system converge toward the largest invariant set $\mathcal{M}$ included in $\mathcal{N} = \{x \text{ s.t. } W(x) = 0\}$

LaSalle invariance principle

The idea is to prove that $W(x) = 0$ is verified only for $x = 0$

Corollary

Assume there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{V}(x) \leq W(x) \leq 0 \qquad \forall x \in \mathbb{R}^n$$

and assume that only the trivial point $x = 0$ remains invariant, then the equilibrium point globally asymptotically stable

Example

Regarding the pendulum example :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a \sin x_1 - b x_2 \end{bmatrix}, \qquad \text{with } a \text{ and } b > 0$$

with the Lyapunov function : $V(x) = a\big(1 - \cos(x_1)\big) + \frac{1}{2} x_2^2$

It was shown that the origin is stable : $\dot{V}(x) = -b x_2^2 \leq 0$

$$\hookrightarrow \dot{V}(x) = 0 \text{ for } x_2 = 0 \text{ and } \forall x_1$$

- It corresponds to the set $\mathcal{N} = \{x \mid x_2 = 0 \text{ and } -2\pi < x_1 < 2\pi\}$
- Assume there is a trajectory in $\mathcal{N}$ such that $x_1 \neq 0 \Rightarrow \dot{x}_2 \neq 0$
- And the trajectory does not belong to $\mathcal{N}$
- Then $x_1 = 0$ and $\mathcal{M} = \{0\} \Rightarrow$ the origin is asymptotically stable

Exercise 1

Consider system :

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -ax_1^3 - bx_2\end{bmatrix}, \qquad \text{with } a \text{ and } b > 0$$

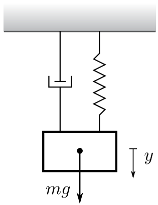
- Determine the equilibrium point
- Show that $V(x) = a\dfrac{x_1^4}{4} + \dfrac{x_2^2}{2}$ is a Lyapunov function
- Apply the LaSalle invariance principle to show the asymptotic stability of the system at the equilibrium point

Solution :

Exercise 2

Mass spring (nonlinear)-damper model :

$$m\ddot{y} = mg - c\dot{y}|\dot{y}| - ky$$



- Give a state space representation such that the origin is an equilibrium point
- Give condition on a and b so that $V(x) = ax_1^2 + bx_2^2$ is a Lyapunov function
- Apply the LaSalle invariance principle to show the asymptotic stability of the system at the equilibrium point

Solution :

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Stability of linear systems

Let's recall some elements on linear systems

$$\dot{x} = Ax \qquad A \in \mathbb{R}^{n\times n}$$

- if $\det(A) \neq 0 \Rightarrow$ unique equilibrium point $x^* = 0$
- if $\det(A) = 0 \Rightarrow$ infinitely many equilibrium point
- and at least one eigenvalue is zero

Theorem

Consider the above linear system, The equilibrium point $x^* = 0$ is :

- **stable** iff $\mathrm{R}_e[\lambda_i] \leq 0$ and for all pure imaginary eigenvalues of algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i \mathbb{I}_n) = n - q_i$
- **asymptotically stable** iff $\mathrm{R}_e[\lambda_i] < 0$
- **unstable** iff there is at least one λ_i is such that $\mathrm{R}_e[\lambda_i] > 0$

Example

The classical transfer function for a DC motor is of the form

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{K}{s(\tau s + 1)} \qquad K > 0,\ \tau > 0$$

A state space representation is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{K}{\tau} \end{bmatrix} u$$

The equilibrium point, for $u = 0$, is parametrized by $x^* = \begin{bmatrix} x_1^* \\ 0 \end{bmatrix}$

(Physical interpretation ?)

$\hookrightarrow$ eigenvalues : 0 and $-\frac{1}{\tau}$ $\quad \Rightarrow \quad$ origin is stable

Lyapunov function candidate for LTI systems

Consider the quadratic Lyapunov function : $V(x) = x^T P x$, with $P > 0$

Time derivative :

$$\begin{aligned}\dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P x \\ &= x^T \left(A^T P + PA \right) x \\ &= -x^T Q x\end{aligned}$$

Equation to be solved : **Lyapunov equation** $A^T P + PA = -Q$

- if $Q > 0$, the equilibrium point is asymptotically stable
- if A is Hurwitz, then P verifying the Lyapunov equation is unique
- the quadratic Lyapunov function is a necessary and sufficient candidate function

Stability condition for LTI systems with Lyapunov mehod

Theorem

A necessary and sufficient condition for a LTI system $\dot{x} = Ax$ to be asymptotically stable is that for any positive definite matrix Q, the unique matrix P solution of the *Lyapunov equation* is positive matrix.

Example :

$$\dot{x} = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} x$$

Let's take $Q = \mathbb{I}$. The Lyapunov equation is :

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Unique solution : $p_{11} = \frac{5}{16}$ and $p_{12} = p_{22} = \frac{1}{16}$

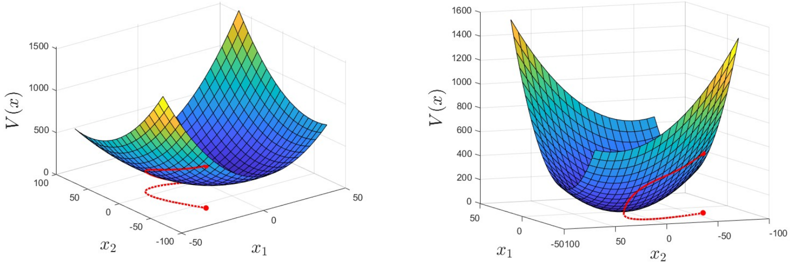
$\hookrightarrow$ P is thus positive definite $\Rightarrow$ origin is asymptotically stable

(note that eigenvalues of A are : -4 and -8)

Plot of the Lyapunov function

$$V(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \frac{5}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And trajectory of x from the initial condition $x_0 = \begin{bmatrix} -20 \\ -60 \end{bmatrix}$



Exercise

Consider the linear system

$$\dot{x} = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix} x$$

- Analyze the stability with the Lyapunov method

Solution :

Stability condition for LTI systems with Lyapunov mehod

Proof of necessity

Assume A is Hurwitz. We want to show that it implies Lyapunov equation holds. Let Q be a positive definite matrix, and let

$$P = \int_0^\infty \exp(A^T t) Q \exp(At)\ dt$$

- The integral exists since A is Hurwitz
- P is positive definite since Q is

Let express the Lyapunov equation (left-hand side) :

$$\begin{aligned} A^T P + PA &= \int_0^\infty A^T \exp(A^T t) Q \exp(At) + \exp(A^T t) Q \exp(At) A\ dt \\ &= \int_0^\infty \frac{d}{dt}\Big(\exp(A^T t) Q \exp(At)\Big)\ dt \\ &= \Big[\exp(A^T t) Q \exp(At)\Big]_0^\infty \ = \ -Q \end{aligned}$$

Indirect Lyapunov method

Idea : use the linear approximation of a nonlinear system to conclude on its stability

$\hookrightarrow$ then use Taylor expansion at the order 1 (linearization) to prove local stability

Theorem

Consider the system $\dot{x} = f(x)$ and its equilibrium x^*. Calculate :

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*}$$

- if $\mathrm{R}_e[\lambda_i] < 0$, then the equilibrium point is locally asymptotically stable
- if there exists an eigenvalue with $\mathrm{R}_e[\lambda_i] > 0$, then the equilibrium point is unstable
- if there exists an eigenvalue with $\mathrm{R}_e[\lambda_i] = 0$, then we cannot conclude

Back to the pendulum example

Nonlinear model :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

Jacobian matrix

$$\frac{\partial f}{\partial x}(x) = A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

Linear models

around $x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \tilde{x}$$

around $x^\star = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} \tilde{x}$$

Setting $m = l = k = 1$ and $g = 10$

$\hookrightarrow$ left system, eigenvalues : $-0.5 \pm 3.12\, i \Rightarrow$ equilibrium locally asympt. stable

$\hookrightarrow$ right system, eigenvalues : -3.7 and $2.7 \Rightarrow$ equilibrium locally unstable

Sommaire

Input to state stability

Consider nonlinear systems of the form

$$\dot{x} = f(t, x, u)$$

- f is a piecewise cont. function w.r.t. time and locally Lipschitz w.r.t. x and u
- the input u is a piecewise continuous function and bounded
- it is assumed that the unforced system

$$\dot{x} = f(t, x, 0)$$

  has an equilibrium point at 0 and is globally asymptotically stable

$\hookrightarrow$ How does the system behave when it is subject to a bounded input u?

Linear systems case

Let's first start with linear systems

$$\dot{x} = Ax + Bu, \qquad A \text{ is assumed to be Hurwitz}$$

for which the solution is known

$$x(t) = e^{At}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)\, d\tau$$

Since A is Hurwitz, $\exists k,\ \lambda$ such that $\|e^{At}\| \leq ke^{-\lambda t}$, we have

$$\|x(t)\| \leq ke^{-\lambda t}\|x_0\| + \int_0^t ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|\, d\tau$$

$$\leq ke^{-\lambda t}\|x_0\| + \frac{k\|B\|}{\lambda} \sup_{0\leq\tau\leq t} \|u(\tau)\|$$

- a bounded input $\Rightarrow$ a state bounded
- the bound on the state is proportional to the bound on the input

What about nonlinear systems ?

Consider this introductory example :

$$\dot{x} = -x + (x^2 + 1)u$$

- Without input, the equilibium point 0 is GAS
- With $u(t) = 1$ (bounded input), the system is unstable
- Differently from linear systems, GAS property $\not\Rightarrow$ ISS

Definitions of comparison functions

Class $\mathcal{K}$ functions

A continuous function α of $[0\,,\,a]$ valued in $[0\,,\,+\infty]$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if $a = +\infty$ and $\lim\limits_{\theta \to +\infty} \alpha(\theta) = +\infty$.

Class $\mathcal{L}$ functions

A continuous function α of $[0\,,\,+\infty]$ valued in $[0\,,\,+\infty]$ is said to be of class $\mathcal{L}$ if it is strictly decreasing and $\lim\limits_{\theta \to +\infty} \alpha(\theta) = 0$.

Class $\mathcal{KL}$ functions

A two argument function is said to be of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ w.r.t. the first argument and of class $\mathcal{L}$ w.r.t. the second one.

Examples of comparison functions

- $\alpha(x) = \tan^{-1}(x)$ is strictly increasing since $\frac{\partial\alpha}{\partial x} = \frac{1}{1+x^2} > 0$. It belongs to $\mathcal{K}$, but not $\mathcal{K}_\infty$ since $\lim_{x\to\infty} \alpha(x) = \frac{\pi}{2}$.

- $\alpha(x) = x^k$, $k > 1$ is strictly increasing since $\frac{\partial\alpha}{\partial x} = kx^{k-1} > 0$. Furthermore, $\lim_{x\to\infty} \alpha(x) = +\infty$, thus α belongs to $\mathcal{K}_\infty$.

- $\beta(x, y) = \dfrac{x}{kxy+1}$, $k > 0$
  - It is strictly increasing in x since $\dfrac{\partial\beta}{\partial x} = \dfrac{1}{(kxy+1)^2} > 0$
  - It is strictly decreasing in y since $\dfrac{\partial\beta}{\partial y} = \dfrac{-kx^2}{(kxy+1)^2} < 0$
  - $\lim_{y\to+\infty} \beta(x, y) = 0$
  - It is function of class $\mathcal{KL}$

- What about the function $\beta(x, y) = x^k e^{-ay}$, $a > 0$, $k > 1$

Lyapunov theorem with comparison functions

We still consider a nonlinear system of the form

$$\dot{x} = f(x)$$

with an equilibrium point at 0

Considering the equilibrium point $x^* = 0$ and a domain $\mathcal{D}$ including 0. Let $V(x) : \mathcal{D} \to \mathbb{R}$ be a C^1 function such that :

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\dot{V}(x) \leq -\alpha_3(\|x\|)$$

then the above system is

- stable if α_1, α_2 are class $\mathcal{K}$ and $\alpha_3 \geq 0$ on $\mathcal{D}$,
- asymptotically stable if α_1, α_2 and α_3 are class $\mathcal{K}_\infty$ functions.

Definition of the Input to State Stability

Definition

A system of the form

$$\dot{x} = f(x, u)$$

is said to be input to state stable (ISS) if and only if it exists a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that for all initial conditions x_0 and all the bounded inputs $u(t)$, the solution $x(t)$ exists for $t \geq 0$ and satisfies :

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma\Big(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\Big)$$

if $u = 0$, then the definition corresponds to the global asymptotic stability of the origin

$\hookrightarrow$ the origin of $\dot{x} = f(x, 0)$ is GAS

ISS means that any bounded input implies a bounded state

Exercise

Consider the system

$$\dot{x} = u - \mathrm{sat}(x)$$

where $\mathrm{sat}(\cdot)$ is the saturation function

$$\mathrm{sat}(x) = \begin{cases} 1 & \text{if} \quad x > 1 \\ x & \text{if} \quad -1 \leq x \leq 1 \\ -1 & \text{if} \quad x < -1 \end{cases}$$

- Show that the system without input ($u = 0$) is GAS.
- Find a particular input u showing that the system is not ISS.

Solution :

Theorem for ISS analysis

The theorem for ISS property is also based on a Lyapunov function.

Consider system : $\dot{x} = f(t, x, u)$

Theorem

Let us consider a function $V \ : \ \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$, a $\mathcal{C}^1$ function such that :

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha_3(\|x\|), \quad \forall \|x\| \geq \rho(\|u\|) > 0$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function and α_3 is a positive definite function, then the system is ISS.

Example 1

Consider the system :

$$\dot{x} = -x^3 + u$$

- The origin of the unforced system is GAS. Shown with the LK : $V(x) = \frac{1}{2}x^2$.
- Using the same LF, its time-derivative along the trajectories of the whole system

$$\dot{V}(x) = -x^4 + xu$$

- Without any change, let's introduce a scalar $\theta \in (0,1)$:

$$\dot{V}(x) = -x^4 + xu + \theta x^4 - \theta x^4 = -(1-\theta)x^4 + x(u - \theta x^3)$$

  We obtain that $\dot{V}(x) \leq -(1-\theta)x^4$ provided that

$$x(u - \theta x^3) < 0 \qquad \text{or equivalently} \qquad |x| > \left(\frac{|u|}{\theta}\right)^3$$

- the system is ISS with $\rho(\|u\|) = \left(\frac{|u|}{\theta}\right)^3$

Example 2

Consider the system :

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2$$

- The origin of the unforced system is GAS. Shown with the LK : $V(x) = \frac{1}{2}x^2$.
- Using the same LF, its time-derivative along the trajectories of the whole system

$$\begin{aligned} \dot{V}(x) = & -x^2 - 2x^4 + x(1 + x^2)u^2 \\ = & -x^4 - x^4 - x^2 + x(1 + x^2)u^2 \\ = & -x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \\ = & -x^4 + (1 + x^2)(-x^2 + xu^2) \end{aligned}$$

  We obtain that $\dot{V}(x) \leq -x^4$ provided that

$$-x^2 + xu^2 < 0 \qquad \text{or equivalently} \qquad |x| > u^2$$

- the system is ISS with $\rho(\|u\|) = |u|^2$