Chapitre 4 : Describing functions

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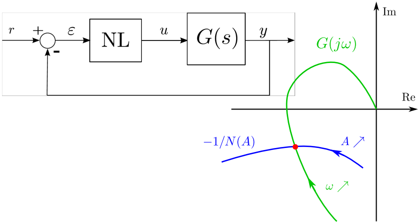

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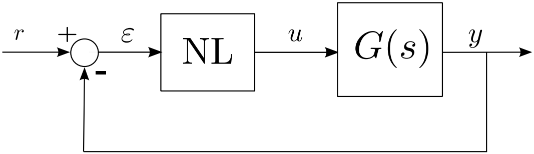
Introduction

- Describing functions method is an extension of harmonic method for some nonlinearities in a closed-loop system
  - $\hookrightarrow$ in french, it is called : *méthode du premier harmonique*
- It approximates a nonlinear element by a “equivalent” linear term
  - $\hookrightarrow$ harmonic linearization
- Method particularly used to predict limit cycle in a closed-loop system

Framework

In this chapter, we only consider closed-loop system of the form

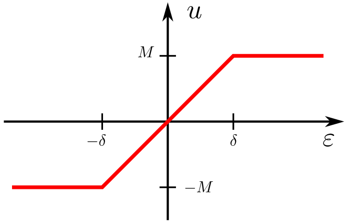


Assumptions :

- The nonlinear element NL is a separable term
- It is time-invariant
- The linear term, $G(s)$, is stable and a low-pass filter type (known as *filtering hypothesis*)

Nonlinear element : example 1

Saturation



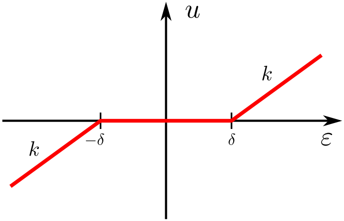
- Linear for $\varepsilon \in [-\delta\ ,\ \delta]$, constant for large values of $|\varepsilon|$.
- Often models actuator limitations

  ↪ power amplifiers, motors, servo-valve for flow control

- Usually caused by limits on component size, properties of materials, available power, mechanical configuration...

Nonlinear element : example 2

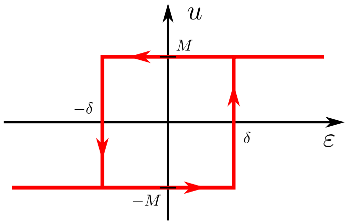
Dead zone



- *u* is zero until the magnitude of the input exceeds some threshold $|\varepsilon| > \delta$.
- Usually characterize actuators (valve, motor...) that are unresponsive to low input signals
- For instance, it models static friction on motor shaft

Nonlinear element : example 3

Hysteresis



- ...
- Examples 1 and 2 are static nonlinearities, also named *memoryless*
  - ↪ the output solely depends on the instantaneous input value
- Hysteresis output depends on the instantaneous and past input values
  - ↪ nonlinear element with *memory*

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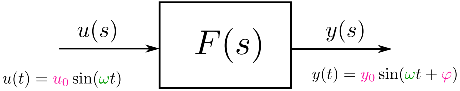
Frequency response

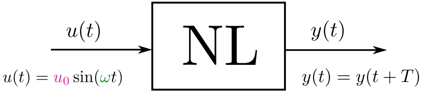
Recall for linear case : The response to a sine function is also a sine function (at steady state)

$\hookrightarrow$ with the same frequency ω but different amplitude and phase shift w.r.t. ω

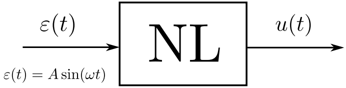


For nonlinear case : The response is a periodic signal (at steady state)

$\hookrightarrow$ $y(t) = y(t+T)$, then a Fourier series expansion can be used



First harmonic approximation



In the closed-loop system, let's assume there is a limit cycle and the oscillating signal is

$$\varepsilon(t) = A\sin(\omega t) \qquad (\equiv -y(t) \quad \text{if reference is 0 })$$

Fourier series expansion of the nonlinear component response

$$u(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

- Fourier coefficients a_0, a_n and b_n are functions of A and ω
- if the nonlinearity is odd, $a_0 = 0$ (often the case)

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First harmonic approximation

The *filtering hypothesis* implies all harmonics are filtered out and only the fundamental component is considered :

$$\begin{aligned} u(t) &\simeq a_1 \cos(\omega t) + b_1 \sin(\omega t) \\ &\simeq M \sin(\omega t + \phi) \\ &\simeq M \; e^{j(\omega t + \phi)} \end{aligned}$$

with

$$a_1(A,\omega) = \frac{\omega}{\pi} \int_{(T)} u(t) \cos(\omega t) \; dt \qquad b_1(A,\omega) = \frac{\omega}{\pi} \int_{(T)} u(t) \sin(\omega t) \; dt$$

$$M(A,\omega) = \sqrt{a_1^2 + b_1^2} \qquad \phi(A,\omega) = \arctan(\frac{a_1}{b_1})$$

Describing function

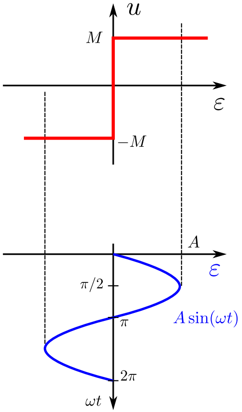
Similarly to linear case, frequency response = ratio sinusoidal output / sinusoidal input

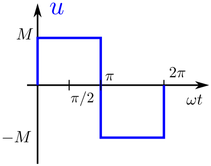
The *describing function* of a nonlinear element is the complex ratio of the fundamental component of the nonlinearity by the input sinusoid

$$N(A,\omega) = \frac{Me^{j(\omega t+\phi)}}{Ae^{j\omega t}} = \frac{M}{A}e^{j\phi} = \frac{1}{A}(b_1 + ja_1)$$

- The approximated frequency response depends on the input amplitude A
  ↪ this operation is called quasi-linearization
- For static nonlinearities, the describing function is independent of ω

Example 1 : relay





First Fourier coefficients :

$$a_1 = 0$$

$$b_1 = \frac{4M}{\pi}$$

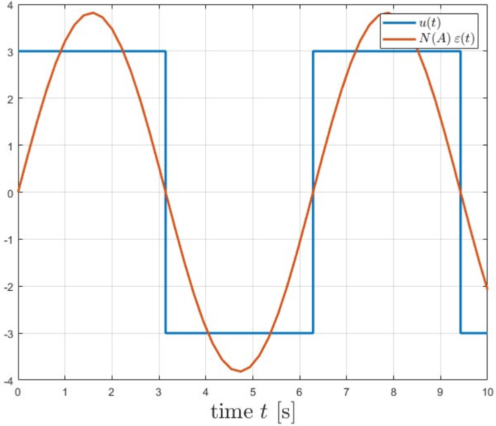
The describing function is

$$N(A) = \frac{4M}{A\pi}$$

Example 1 : relay

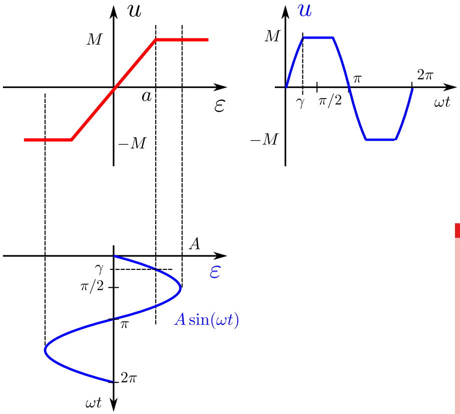
Simulation for : $\omega = 1$, $A = 2$ and $M = 3$

$\hookrightarrow$ equivalent gain $N(A) = 1.909$



◊ Keeping in mind that u is then filtered by a low-pass type transfer function

Example 2 : saturation



First Fourier coefficients :

$$a_1 = 0$$

$$b_1 = \frac{2M}{\pi}\left(\frac{A}{a}\arcsin\left(\frac{a}{A}\right) + \sqrt{1 - \frac{a^2}{A^2}}\right)$$

If $a < A$, the describing function is

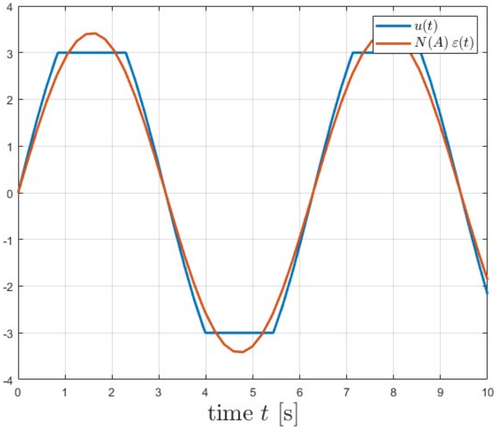
$$N(A) = \frac{2M}{\pi}\left(\frac{1}{a}\arcsin\left(\frac{a}{A}\right) + \frac{1}{A}\sqrt{1 - \frac{a^2}{A^2}}\right)$$

If $a > A$, no saturation and $N(A) = M/a$

Example 2 : saturation

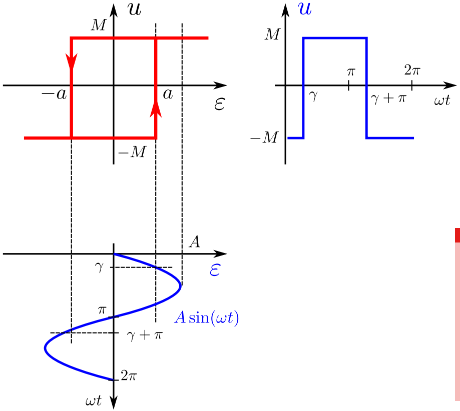
Simulation for : $\omega = 1$, $A = 2$, $M = 3$ and $a = 1.5$

$\hookrightarrow$ equivalent gain $N(A) = 1.711$



◊ Keeping in mind that u is then filtered by a low-pass type transfer function

Example 3 : hysteresis



First Fourier coefficients :

$$a_1 = -\frac{4M}{\pi}\frac{a}{A}$$

$$b_1 = \frac{4M}{\pi}\sqrt{1 - \frac{a^2}{A^2}}$$

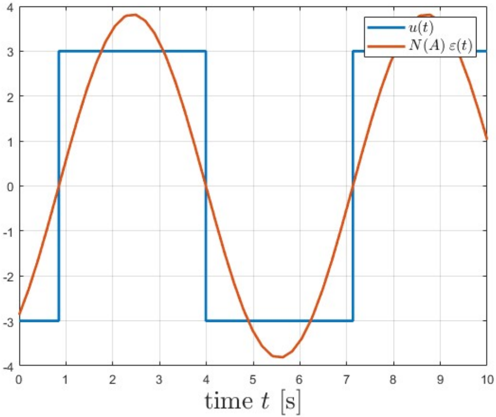
If $a < A$, the describing function is

$$N(A) = \frac{4M}{A\pi}\left(\sqrt{1 - \frac{a^2}{A^2}} - j\frac{a}{A}\right)$$

Example 3 : hysteresis

Simulation for : $\omega = 1$, $A = 2$, $M = 3$ and $a = 1.5$

$$\hookrightarrow \text{equivalent gain } N(A) = 1.26 - 1.43i, \text{ soit } \begin{cases} M = 1.909 \\ \phi = -0.848 \end{cases}$$



◊ Keeping in mind that u is then filtered by a low-pass type transfer function

Computing describing functions

Different methods to compute a describing function of a nonlinear element

$$u = f(\varepsilon)$$

- Analytical calculation. when the nonlinear characteristic is known, explicit and simple enough to calculate the integrals (a_1 and b_1) ; the nonlinearity could also be approximated by piecewise linear functions. Result is an analytical expression of $N(A, \omega)$.

- Numerical integration. when the nonlinear characteristic is given by a graph / table of values, integrals numerically computed with a discrete sums of surface over small intervals (numerical algorithm). Result is a plot of N w.r.t A and ω.

- Experimental evaluation. Interesting when no information about the nonlinearity (or too complex), but can be isolated and excited with sinusoidal inputs for various A and ω ; compute the ratio of amplitude and phase shift with the output. Result is a plot of N w.r.t A and ω.

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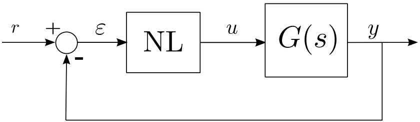
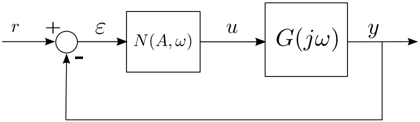
Closed-loop system analysis

Let's go back to closed-loop system (with $r = 0$)



Assuming the system is oscillating, the closed-loop can be approximated by



◊ The output must satisfy the relationship : $y = G(j\omega)N(A,\omega)(-y)$

Existence of limit cycles

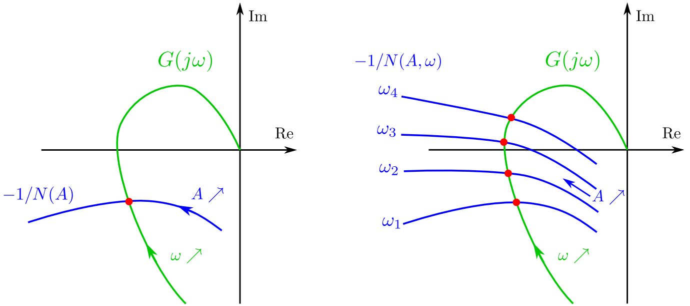
It implies that

$$G(j\omega)N(A,\omega)+1=0 \qquad \Leftrightarrow \qquad G(j\omega)=-\frac{1}{N(A,\omega)}$$

- If some solutions exist, there is (are) limit cycle(s) with amplitude A and frequency ω (approximately)
- if not, there is no limit cycle
- It is 2 nonlinear equations with 2 variables (A and ω)
- May be very difficult to solve analytically for high-order systems
- Usually, graphical approach
  - ↪ plot $G(j\omega)$ and $-1/N(A,\omega)$ in the complex plane
  - ↪ find the intersection points

Existence of limit cycles : graphical method

Illustration of the method when the describing function depends only on the amplitude A (left) and on both amplitude A and frequency ω (right)

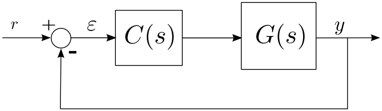


Note that in both cases there could be several intersection points

Stability of limit cycles

Previous slides were about **detecting existence** of limit cycles. What about their **stability** ?

Before that, let's recall the Nyquist criterion



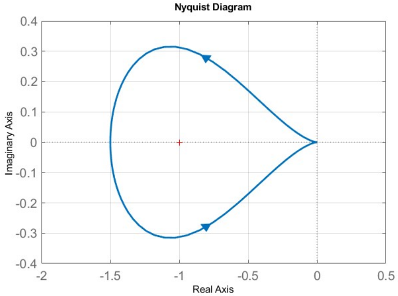
Characteristic equ. of the closed-loop system : $1 + C(s)G(s) = 0$ (or $CG = -1$)

Nyquist criterion

Procedure :

- Draw in the complex plane $C(s)G(s)$, s following the Nyquist path
- Determine the number N of clockwise encirclement around the point $(-1, 0)$
- Determine the number P of unstable poles of $C(s)G(s)$
- Then $Z = N + P$ is the number of unstable poles of the closed-loop system

Example : with $C(s) = 1$ and $G(s) = \dfrac{3}{(s+2)(s-1)}$



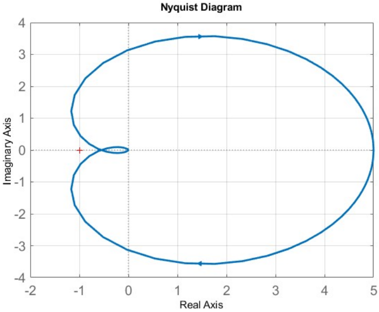
- Number of clockwise encirclement around the point $(-1, 0)$: $N = -1$
- Number of unstable poles of $C(s)G(s)$: $P = 1$
- Then, there is $Z = N + P = 0$ unstable pole for the closed-loop system

  ↪ closed-loop system stable

Simplified version : when $C(s)G(s)$ has no unstable pole (critère de Revers)

$\Rightarrow$ no encirclement around the point $(-1, 0) \rightarrow$ closed-loop system stable

Example with $C(s)G(s) = \dfrac{10}{(s+1)^2(s+2)}$

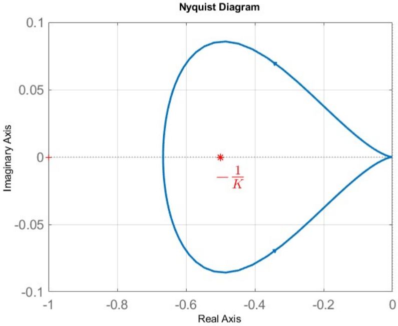


Simple extension : as a function of a tunable gain K in the loop

The characteristic equation : $1 + KC(s)G(s) = 0$ or $C(s)G(s) = -\dfrac{1}{K}$

$\Rightarrow$ check the encirclement around the point $(-1/K, 0)$

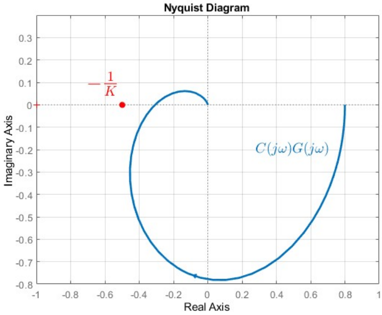
Example with $C(s)G(s) = \dfrac{1}{(s+1.5)(s-1)}$ and $K = 2$



Actually system stable $\forall K > 1.5$

Another Example with $C(s)G(s) = \dfrac{1}{s^3 + 2s^2 + 2.25s + 1.25}$ and $K = 2$

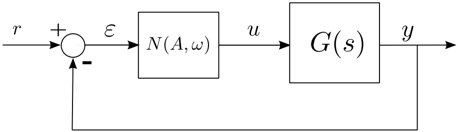
(poles = -1 and $-0.5 \pm j$)



Actually system stable for $-1.25 < K < 3.25$

Stability of limit cycles

Back to closed-loop system with a describing function :



Characteristic equation : $1 + N(A,\omega)G(s) = 0$ or $G(s) = -\frac{1}{N(A,\omega)}$

- Check encirclement around the point $\left(R_e[-\frac{1}{N}], I_m[-\frac{1}{N}]\right)$
- By assumption $G(s)$ is stable $\rightarrow$ no unstable pole
- Check if critical points is on left or right of $G(j\omega)$ locus when $\omega \nearrow$

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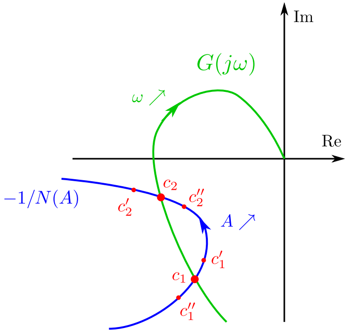
Graphical analysis

Consider a system where two limit cycles are predicted



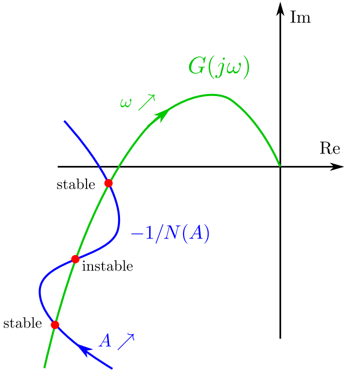
- At c_1, there is a limit cycle with an amplitude A_1 and a frequency ω_1
  - if a slight disturbance increases A ; we move to c_1' ; the system is unstable ; the amplitude continues to increase ; we move along curve $-1/N(A)$ toward c_2
  - if a slight disturbance decreases A ; we move to c_1'' ; the system is stable ; the amplitude continues to decrease ; we move along curve $-1/N(A)$ toward 0
  - the limit cycle is unstable
- At c_2, there is a limit cycle with an amplitude A_2 and a frequency ω_2
  - if a slight disturbance increases A ; we move to c_2' ; the system is stable ; the amplitude decreases ; we move back toward c_2
  - if a slight disturbance decreases A ; we move to c_2'' ; the system is unstable ; the amplitude increases ; we move back toward c_2
  - the limit cycle is stable

Stability condition (graphical)

Loeb criterion

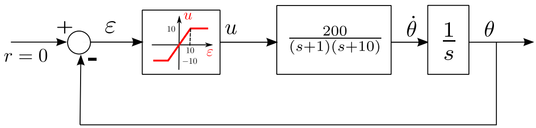
A limit cycle of amplitude A_0 and frequency ω_0 is stable if the intersection point is such that along the nyquist plot of $G(j\omega)$ as ω increases, the direction of increasing A along the critical curve $-1/N(A)$ is toward the left.

Example :



Example

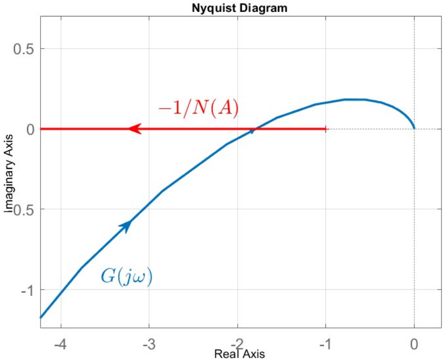
Simple control of a DC motor with a saturation



- What is the describing function of the nonlinearity ?
- Show that a limit cycle exists.
- What would be the approximated amplitude and frequency of the self-oscillations ?
- Is the limit cycle stable ?

Example

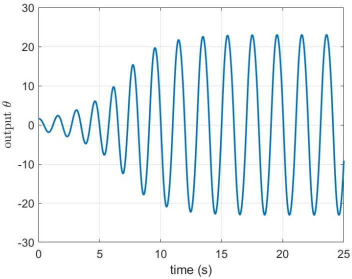
- Describing function : $N(A) = \frac{2}{\pi} \arcsin\left(\frac{10}{A}\right) + \frac{20}{A\pi}\sqrt{1 - \frac{100}{A^2}}$ (if $A > 10$)
- Nyquist plot of $G(j\omega)$ and $-1/N(A)$; note that $N(10) = 1$ and $N(+\infty) = 0$

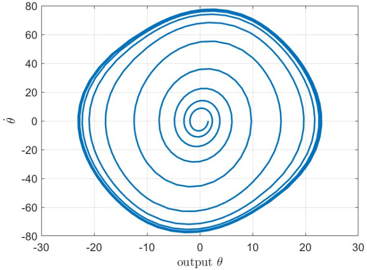


- From the plot, $\omega = 3.22\ rad/s$ ($T = 1.95\ s$) and $A = 22.1$
- The limit cycle is stable

Example

Simulation of the closed loop : ouput (left) and phase plane (right)





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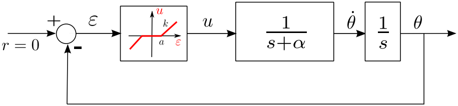
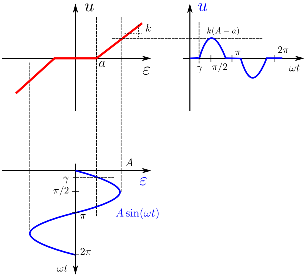
Exercise

Simple control of a DC motor with a dead zone





- What is the describing function of the nonlinearity ?
- Draw a sketch of the nyquist plot of $G(j\omega)$ and the critical locus $-1/N(A)$.
- Does a limit cycle exists ?

Solution