ANNEXE C – Modèle mathématique du robot lego NXT

tiré de la notice NXTway-GS Model-Based Design - Control of self-balancing two-wheeled robot built with LEGO Mindstorms NXT, Yorihisa Yamamoto.

3 NXTway-GS Modeling

This chapter describes mathematical model and motion equations of NXTway-GS.

3.1 Two-Wheeled Inverted Pendulum Model

NXTway-GS can be considered as a two wheeled inverted pendulum model shown in Figure 3-1.

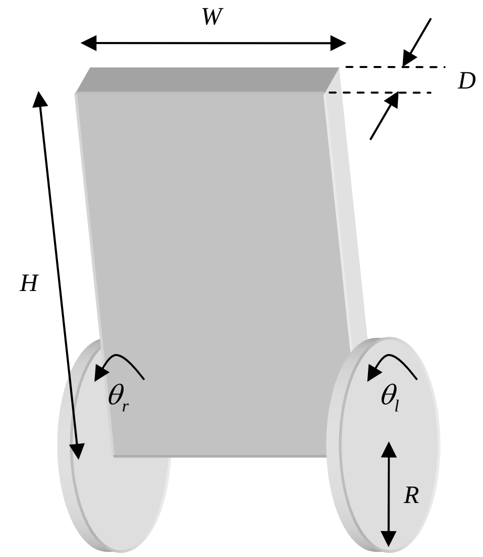
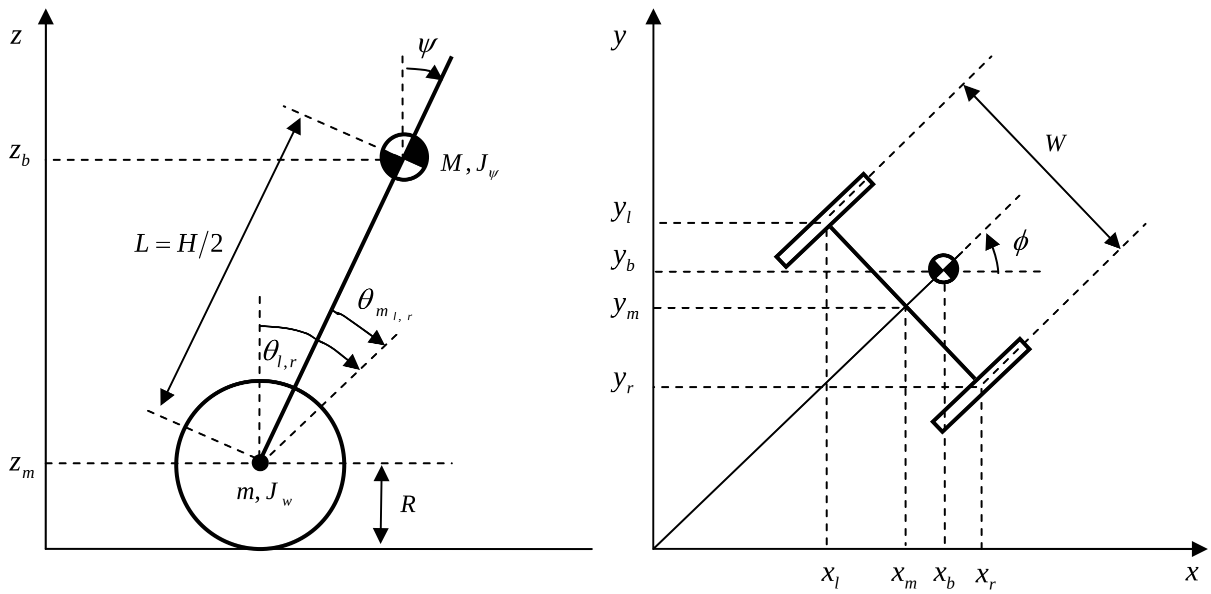


Figure 3-1 Two-wheeled inverted pendulum

Figure 3-2 shows side view and plane view of the two wheeled inverted pendulum. The coordinate system used in 3.2 Motion Equations of Two-Wheeled Inverted Pendulum is described in Figure 3-2.



ψ : body pitch angle $\theta_{l,r}$: wheel angle (l, r indicates left and right) $\theta_{m_{l,r}}$: DC motor angle

Figure 3-2 Side view and plane view of two-wheeled inverted pendulum

Physical parameters of NXTway-GS are the following.

$g = 9.81$	$[m/\sec^2]$	:	Gravity acceleration
$m = 0.03$	$[kg]$	:	Wheel weight
$R = 0.04$	$[m]$	:	Wheel radius
$J_W = mR^2/2$	$[kgm^2]$	:	Wheel inertia moment
$M = 0.6$	$[kg]$	:	Body weight
$W = 0.14$	$[m]$	:	Body width
$D = 0.04$	$[m]$	:	Body depth
$H = 0.144$	$[m]$	:	Body height
$L = H/2$	$[m]$	:	Distance of the center of mass from the wheel axle
$J_\psi = ML^2/3$	$[kgm^2]$	:	Body pitch inertia moment
$J_\phi = M\left(W^2 + D^2\right)/12$	$[kgm^2]$	:	Body yaw inertia moment
$J_m = 1\times 10^{-5}$	$[kgm^2]$	:	DC motor inertia moment
$R_m = 6.69$	$[\Omega]$	:	DC motor resistance
$K_b = 0.468$	$[V\sec/rad]$	:	DC motor back EMF constant
$K_t = 0.317$	$[Nm/A]$	:	DC motor torque constant
$n = 1$		:	Gear ratio
$f_m = 0.0022$		:	Friction coefficient between body and DC motor
$f_W = 0$		:	Friction coefficient between wheel and floor.

- We use the values described in reference [2] for R_m, K_b, K_t.
- We use the values that seems to be appropriate for J_m, n, f_m, f_W, because it is difficult to measure.

3.2 Motion Equations of Two-Wheeled Inverted Pendulum

We can derive motion equations of two-wheeled inverted pendulum by the Lagrangian method based on the coordinate system in Figure 3-2. If the direction of two-wheeled inverted pendulum is x-axis positive direction at $t = 0$, each coordinates are given as the following.

$$(\theta, \quad \phi) = \left(\frac{1}{2}(\theta_l + \theta_r) \quad \frac{R}{W}(\theta_r - \theta_l)\right) \tag{3.1}$$

$$(x_m, \quad y_m, \quad z_m) = \left(\int \dot{x}_m dt, \quad \int \dot{y}_m dt, \quad R\right), \quad (\dot{x}_m, \quad \dot{y}_m) = \left(R\dot{\theta}\cos\phi \quad R\dot{\theta}\sin\phi\right) \tag{3.2}$$

$$(x_l, \quad y_l, \quad z_l) = \left(x_m - \frac{W}{2}\sin\phi, \quad y_m + \frac{W}{2}\cos\phi, \quad z_m\right) \tag{3.3}$$

$$(x_r, \quad y_r, \quad z_r) = \left(x_m + \frac{W}{2}\sin\phi, \quad y_m - \frac{W}{2}\cos\phi, \quad z_m\right) \tag{3.4}$$

$$(x_b, \quad y_b, \quad z_b) = (x_m + L\sin\psi\cos\phi, \quad y_m + L\sin\psi\sin\phi, \quad z_m + L\cos\psi) \tag{3.5}$$

The translational kinetic energy T_1, the rotational kinetic energy T_2, the potential energy U are

$$T_1 = \frac{1}{2}m\left(\dot{x}_l^{\,2} + \dot{y}_l^{\,2} + \dot{z}_l^{\,2}\right) + \frac{1}{2}m\left(\dot{x}_r^{\,2} + \dot{y}_r^{\,2} + \dot{z}_r^{\,2}\right) + \frac{1}{2}M\left(\dot{x}_b^{\,2} + \dot{y}_b^{\,2} + \dot{z}_b^{\,2}\right) \tag{3.6}$$

$$T_2 = \frac{1}{2}J_w\dot{\theta}_l^{\,2} + \frac{1}{2}J_w\dot{\theta}_r^{\,2} + \frac{1}{2}J_\psi\dot{\psi}^2 + \frac{1}{2}J_\phi\dot{\phi}^2 + \frac{1}{2}n^2J_m(\dot{\theta}_l - \dot{\psi})^2 + \frac{1}{2}n^2J_m(\dot{\theta}_r - \dot{\psi})^2 \tag{3.7}$$

$$U = mgz_l + mgz_r + Mgz_b \tag{3.8}$$

The fifth and sixth term in T_2 are rotation kinetic energy of an armature in left and right DC motor. The Lagrangian L has the following expression.

$$L = T_1 + T_2 - U \tag{3.9}$$

We use the following variables as the generalized coordinates.

θ : Average angle of left and right wheel
ψ : Body pitch angle
ϕ : Body yaw angle

Lagrange equations are the following

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F_\theta \tag{3.10}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = F_\psi \tag{3.11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = F_\phi \tag{3.12}$$

We derive the following equations by evaluating Eqs. (3.10) - (3.12).

$$\left[(2m+M)R^2 + 2J_w + 2n^2J_m\right]\ddot{\theta} + \left(MLR\cos\psi - 2n^2J_m\right)\ddot{\psi} - MLR\dot{\psi}^2\sin\psi = F_\theta \tag{3.13}$$

$$\left(MLR\cos\psi - 2n^2J_m\right)\ddot{\theta} + \left(ML^2 + J_\psi + 2n^2J_m\right)\ddot{\psi} - MgL\sin\psi - ML^2\dot{\phi}^2\sin\psi\cos\psi = F_\psi \tag{3.14}$$

$$\left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2R^2}\left(J_w + n^2J_m\right) + ML^2\sin^2\psi\right]\ddot{\phi} + 2ML^2\dot{\psi}\dot{\phi}\sin\psi\cos\psi = F_\phi \tag{3.15}$$

In consideration of DC motor torque and viscous friction, the generalized forces are given as the following

$$\left(F_\theta, \quad F_\psi, \quad F_\phi\right)=\left(F_l + F_r, \quad F_\psi, \quad \frac{W}{2R}\left(F_r - F_l\right)\right) \tag{3.16}$$

$$F_l = nK_t i_l + f_m(\dot{\psi} - \dot{\theta}_l) - f_w \dot{\theta}_l \tag{3.17}$$

$$F_r = nK_t i_r + f_m(\dot{\psi} - \dot{\theta}_r) - f_w \dot{\theta}_r \tag{3.18}$$

$$F_\psi = -nK_t i_l - nK_t i_r - f_m(\dot{\psi} - \dot{\theta}_l) - f_m(\dot{\psi} - \dot{\theta}_r) \tag{3.19}$$

where $i_{l,r}$ is the DC motor current.

We cannot use the DC motor current directly in order to control it because it is based on PWM (voltage) control. Therefore, we evaluate the relation between current $i_{l,r}$ and voltage $v_{l,r}$ using DC motor equation. If the friction inside the motor is negligible, the DC motor equation is generally as follows

$$L_m \dot{i}_{l,r} = v_{l,r} + K_b(\dot{\psi} - \dot{\theta}_{l,r}) - R_m i_{l,r} \tag{3.20}$$

Here we consider that the motor inductance is negligible and is approximated as zero. Therefore the current is

$$i_{l,r} = \frac{v_{l,r} + K_b(\dot{\psi} - \dot{\theta}_{l,r})}{R_m} \tag{3.21}$$

From Eq.(3.21), the generalized force can be expressed using the motor voltage.

$$F_\theta = \alpha\left(v_l + v_r\right) - 2\left(\beta + f_w\right)\dot{\theta} + 2\beta\dot{\psi} \tag{3.22}$$

$$F_\psi = -\alpha\left(v_l + v_r\right) + 2\beta\dot{\theta} - 2\beta\dot{\psi} \tag{3.23}$$

$$F_\phi = \frac{W}{2R}\alpha\left(v_r - v_l\right) - \frac{W^2}{2R^2}\left(\beta + f_w\right)\dot{\phi} \tag{3.24}$$

$$\alpha = \frac{nK_t}{R_m}, \quad \beta = \frac{nK_t K_b}{R_m} + f_m \tag{3.25}$$

3.3 State Equations of Two-Wheeled Inverted Pendulum

We can derive state equations based on modern control theory by linearizing motion equations at a balance point of NXTway-GS. It means that we consider the limit $\psi \to 0$ ($\sin\psi \to \psi$, $\cos\psi \to 1$) and neglect the second order term like $\dot{\psi}^2$. The motion equations (3.13) – (3.15) are approximated as the following

$$\left[(2m+M)R^2 + 2J_w + 2n^2 J_m\right]\ddot{\theta} + \left(MLR - 2n^2 J_m\right)\ddot{\psi} = F_\theta \tag{3.26}$$

$$\left(MLR - 2n^2 J_m\right)\ddot{\theta} + \left(ML^2 + J_\psi + 2n^2 J_m\right)\ddot{\psi} - MgL\psi = F_\psi \tag{3.27}$$

$$\left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2R^2}\left(J_w + n^2 J_m\right)\right]\ddot{\phi} = F_\phi \tag{3.28}$$

Eq. (3.26) and Eq. (3.27) has θ and ψ, Eq. (3.28) has ϕ only. These equations can be expressed in the form

$$E\begin{bmatrix}\ddot{\theta}\\ \ddot{\psi}\end{bmatrix} + F\begin{bmatrix}\dot{\theta}\\ \dot{\psi}\end{bmatrix} + G\begin{bmatrix}\theta\\ \psi\end{bmatrix} = H\begin{bmatrix}v_l\\ v_r\end{bmatrix} \tag{3.29}$$

$$E = \begin{bmatrix}(2m+M)R^2 + 2J_w + 2n^2 J_m & MLR - 2n^2 J_m\\ MLR - 2n^2 J_m & ML^2 + J_\psi + 2n^2 J_m\end{bmatrix}$$

$$F = 2\begin{bmatrix}\beta + f_w & -\beta\\ -\beta & \beta\end{bmatrix}$$

$$G = \begin{bmatrix}0 & 0\\ 0 & -MgL\end{bmatrix}$$

$$H = \begin{bmatrix}\alpha & \alpha\\ -\alpha & -\alpha\end{bmatrix}$$

$$I\ddot{\phi} + J\dot{\phi} = K(v_r - v_l) \tag{3.30}$$

$$I = \frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2R^2}\left(J_w + n^2 J_m\right)$$

$$J = \frac{W^2}{2R^2}\left(\beta + f_w\right)$$

$$K = \frac{W}{2R}\alpha$$

Here we consider the following variables $\mathbf{x_1}, \mathbf{x_2}$ as state, and $\mathbf{u}$ as input. $\mathbf{x}^T$ indicates transpose of $\mathbf{x}$.

$$\mathbf{x_1} = \left[\theta, \quad \psi, \quad \dot{\theta}, \quad \dot{\psi}\right]^T, \quad \mathbf{x_2} = \left[\phi, \quad \dot{\phi}\right]^T, \quad \mathbf{u} = \left[v_l, \quad v_r\right]^T \tag{3.31}$$

Consequently, we can derive state equations of two-wheeled inverted pendulum from Eq. (3.29) and Eq. (3.30).

$$\dot{\mathbf{x}}_\mathbf{1} = A_1\mathbf{x_1} + B_1\mathbf{u} \tag{3.32}$$

$$\dot{\mathbf{x}}_\mathbf{2} = A_2\mathbf{x_2} + B_2\mathbf{u} \tag{3.33}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & A_1(3,2) & A_1(3,3) & A_1(3,4) \\ 0 & A_1(4,2) & A_1(4,3) & A_1(4,4) \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ B_1(3) & B_1(3) \\ B_1(4) & B_1(4) \end{bmatrix} \tag{3.34}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -J/I \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ -K/I & K/I \end{bmatrix} \tag{3.35}$$

$$\begin{aligned}
A_1(3,2) &= -gMLE(1,2)/\det(E) \\
A_1(4,2) &= gMLE(1,1)/\det(E) \\
A_1(3,3) &= -2\left[(\beta + f_w)E(2,2) + \beta E(1,2)\right]/\det(E) \\
A_1(4,3) &= 2\left[(\beta + f_w)E(1,2) + \beta E(1,1)\right]/\det(E) \\
A_1(3,4) &= 2\beta\left[E(2,2) + E(1,2)\right]/\det(E) \\
A_1(4,4) &= -2\beta\left[E(1,1) + E(1,2)\right]/\det(E) \\
B_1(3) &= \alpha\left[E(2,2) + E(1,2)\right]/\det(E) \\
B_1(4) &= -\alpha\left[E(1,1) + E(1,2)\right]/\det(E) \\
\det(E) &= E(1,1)E(2,2) - E(1,2)^2
\end{aligned}$$