An analytical formula for elastic–plastic instability of large oil storage tanks

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1. Introduction

Large oil storage tanks are widely used in the petrochemical industry and crude oil reserve bases. However, with large-scale oil tanks, security problems are becoming more and more prominent and have been the focus of research worldwide.

The failure mode of a large oil storage tank when it suffers an earthquake [1] is shown in Fig. 1. This is called elephant foot buckling because of its axisymmetric elephant foot shape. Under the action of a seismic wave, the pulse and convective pressure formed by the liquid in the tank lead to a rapid increase of dynamic fluid pressure inside the tank [2–4], and local yielding. Therefore, elephant foot buckling is a type of elastic–plastic buckling under the interaction of both circumferential tensile stress and axial compressive stress.

Elephant foot buckling can result in oil spills, fires, explosion accidents and even disastrous ecological disasters. In order to prevent it, much research [5–11] has been performed in the past few decades leading to results adopted by advanced steel shell design standards [12–14].

Previous research has focused on the effect caused by high hydraulic pressure, and the material is regarded as isotropic ideal elastic–plastic without considering its hardening. The critical stress of the tank wall is fitted to an equation including internal pressure and the size of the tank wall. However, there is a large difference between the tensile strength and yield strength of the tank wall’s material. Furthermore, in order to select the right steel to manufacture and construct an oil tank, it is necessary to know the effect of material properties on the buckling strength of the tank wall when elastic–plastic buckling occurs.

The formulas for tank wall critical buckling stress in the present oil tank design standards such as API 650 [15], JIS B 8501 [16] and GB 50341 [17] are based on elastic stability theory. However, elephant foot buckling is elastic–plastic buckling, therefore a correction should be made for plasticity.

This paper focuses on how the plastic properties of the material of the tank wall influence the buckling critical stress. A simple model for buckling analysis of an oil tank wall is presented. Then the buckling critical stress formula of the simplified model is obtained after the analysis of elastic–plastic buckling which is carried out by J2 plastic flow theory. Furthermore, a new corrected formula for calculating critical instability stress of the tank wall is obtained by introducing a plasticity influence coefficient. Numerical calculations and comparisons with current formulas and Rotter’s semi-empirical formula are carried out. The reason why material plasticity has a significant influence on oil tank elastic–plastic buckling is also analyzed.

2. The elastic–plastic buckling critical stress of large oil storage tank wall

2.1. A simplified model used to analyze the plastic buckling of large oil storage tank under high hydraulic pressure

A large oil storage tank is a thin-walled structure, of which radius and thickness ratio \( R/t \) is generally larger than 1000. The tank
wall is subjected to a linearly distributed hydraulic pressure $p$ as shown in Fig. 2, and in order to save material and obey the equal strength design concept, the ideal tank wall thickness should be calculated by Eq. (1) according to the non-moment shell theory [18].

$$t = \frac{\rho g (L - x) R}{[\sigma]}$$  \hspace{1cm} (1)

where $t$ is the thickness of the tank wall; $\rho$ is the density of oil; $g$ is gravity acceleration; $L$ and $R$ denote the height of liquid level and tank wall radius, respectively; $x$ denotes axial coordinate; $[\sigma]$ is allowable stress.

Thus, the circumferential membrane stress $\sigma_{\theta}$ in the tank wall is:

$$\sigma_{\theta} = \frac{\rho g (L - x) R}{t} = [\sigma]$$  \hspace{1cm} (2)

Thus, $\sigma_{\theta}$ is a constant, and the materials are used efficiently. However, in practice, large steel shells with linear thickness are difficult to manufacture and construct. Therefore the tank usually consists of several welded cylinders with different wall thickness as shown in Fig. 2. The wall thickness varies stepwise from the bottom

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**Nomenclature**

- $a_{ij}$: elastic-plastic instability coefficients
- $A_m$: undetermined constants
- $b_{ij}$: elastic-plastic instability coefficients
- $B_m$: undetermined constants
- $C$: constant
- $C_m$: undetermined constants
- $D$: shell diameter
- $E$: elasticity modulus
- $E_t$: tangent modulus
- $E_s$: secant modulus
- $J_2$: $J_2$ stress function
- $K$: material parameter
- $L$: height of liquid level
- $m$: number of half-waves along the shell length at buckling
- $M_p$: plasticity correction coefficient
- $M_{x}, M_{y}$: moment resultants
- $\delta M_{x}, \delta M_{y}$: moment increments
- $n$: material parameter
- $N$: axial compression
- $N_{x}, N_{y}, N_{xy}, N_{xz}, N_{yz}$: force resultants
- $\delta N_{x}, \delta N_{y}, \delta N_{xy}, \delta N_{xz}, \delta N_{yz}$: force increments
- $p, P$: hydraulic pressure, shell radius
- $s$: ratio of radius and thickness
- $S_{ij}$: stress deviator
- $\delta S_{ij}$: stress deviator increment
- $t$: shell thickness
- $t_i$: thickness of the $i$-th shell course of the tank
- $u$: axial displacement
- $v$: circumferential displacement
- $w$: radial displacement
- $\delta u$: axial displacement increment
- $\delta v$: circumferential displacement increment
- $\delta w$: radial displacement increment
- $x$: axial coordinate
- $y$: circumferential coordinate
- $z$: radial coordinate
- $\sigma$: tensile stress
- $\sigma_{eq}$, $\sigma_{y}$: normal stresses
- $\sigma_{sbp}$, $\sigma_{sy0}$: pre-buckling non-moment stresses
- $\sigma_{ct}$: buckling critical stress
- $\sigma_{el}$: elastic classical buckling critical stress
- $\sigma_{y}$: material yield strength
- $\sigma_{\theta}$: circumferential stress
- $[\sigma]$: allowable stress
- $\sigma_{ij}$: stress tensor
- $\delta \sigma_{ij}$: stress tensor increment
- $\varepsilon$: tensile strain
- $\varepsilon_x$, $\varepsilon_y$: normal strains
- $\varepsilon_{ij}$: strain tensor
- $\delta \varepsilon_{ij}$: strain tensor increment
- $\gamma_{xy}$: shear strain
- $\kappa_x$, $\kappa_y$: curvatures of middle surface
- $\kappa_{xy}$: torsional curvature of middle surface
- $\alpha_{x}$: dimensionless stress
- $\alpha_{pp}$: plastic reduction coefficient
- $\rho$: density
- $g$: gravity acceleration

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**Fig. 1.** Appearance of elephant foot buckling in large oil storage tank.

**Fig. 2.** Large oil storage tank with stepwise variable thickness.
to the top. In order to obtain an approximately uniform circumferential membrane stress, current oil tank design standards provide two methods to calculate the thickness, which are the fixed design point method [15–17] and the variable design point method (Appendix K in reference [15]). Theoretical calculation and practical stress test [19,20] shows that, in the static state, circumferential stress and axial stress in the tank wall designed by standards fluctuate. The current standards consider the oil tank as a cylindrical shell with uniform thickness when checking stability of the tank, but the effect of thickness variation is regarded as an uncertain factor to be taken into account by a safety coefficient. Moreover, buckling critical stress formulas adopted by current standards are also derived from the results for an equal-thickness shell [21–23].

Based on the analyses above, a simplified model for the buckling analysis of an oil tank is presented shown in Fig. 3. It is an equal-wall thickness cylindrical shell which is simply supported at both ends and loaded with evenly distributed hydraulic pressure \( P \) and uniform axial compression \( N \). The effect of variable wall thickness on buckling critical stress will be considered by introducing a safety coefficient. Under the effect of a seismic wave, the dynamic fluid pressure [2,4] is not distributed evenly. From the perspective of convenience and safety, hydraulic pressure is also assumed to be uniform, at a value equal to the maximum possible hydraulic pressure acting on the tank wall. Axial compression \( N \) includes tank weight, seismic load, snow load and so on. Detailed calculations of hydraulic pressure and axial compression are given in relevant standards [15–17]. It is pointed out that the thickness of the model equals that of the oil tank's first course, and other geometry parameters are the same as those of an actual tank.

Calculation results for circumferential membrane stresses of an actual oil tank and the simplified model are shown in Fig. 4. Stresses of the actual oil tank come from reference [20] and those of the equivalent model are obtained by shell theory [18]. The theoretical circumferential membrane stress of the proposed model is a good approximation to that of the actual oil tank and the presented boundary conditions approach the actual ones [20,24,25]. Neglecting the edge effect, the shell is in a state of plane stress before buckling, and in the middle surface there only exist an axial stress \( \sigma_0 = -N/2\pi R \) and a circumferential stress \( \sigma_0 = PR/t \). Elastic buckling of the model shown in Fig. 3 has been studied [21,26–30], and the research showed that when the internal pressure is increased, the imperfection sensitivity of the cylindrical shell decreases leading to the increase of the buckling strength, and the buckling mode tends to be axisymmetric. Lo et al. [21] carried out an experimental study of this problem, and the results showed that the buckling critical load increased with increase of internal pressure at the beginning of the experiment, and remained a constant after internal pressure reached a certain value. This result was adopted by API 650. Hutchinson [27] presented a theoretical study on the elastic buckling of thin shells under combined internal pressure and axial compression. Fung [28] also experimentally studied the buckling of a cylindrical shell under the interaction of axial load and internal pressure and the results showed that the buckling mode had a diamond mode when the value of internal pressure was zero, and changed into a square mode when the value of internal pressure increased to a certain value. When internal pressure was large enough, a ribbon ripple in the circumferential direction would occur and the buckling mode became axisymmetric. Rotter and Teng [29,30] also investigated the elastic buckling of imperfect cylindrical shells under combined axial compression and internal pressure by the finite element method, and the results confirmed previous conclusions.

Rotter [7] firstly used the model shown in Fig. 3 to study elephant foot buckling of equal-wall thickness thin-walled cylindrical shells such as grain silos and liquid tanks which suffered combined high internal pressure and axial compression by finite element analysis. In his research, the material was assumed to be ideal elastic—plastic, and a semi-empirical formula was determined.

$$\lambda_x = \frac{\sigma_s}{\sigma_{cl}} \quad s = \frac{1}{400} \frac{R}{t} \quad P = \frac{PR}{\sigma_{cl}} \quad \alpha_{xpp} = \left(1 - \frac{P^2}{\lambda_x^2}\right)$$

$$\times \left(1 - \frac{1}{1.12 + s} \right) \left(\frac{s^2 + 1.21\lambda_x^2}{s(s + 1)}\right) \sigma_{cr} = \alpha_{xpp} \sigma_{cl} \quad (3)$$

where \( \sigma_{cr} \) is the buckling critical stress; \( \sigma_{cl} \) is the elastic classical buckling critical stress \( s_{cl} = 0.605Eti/R \); \( \lambda_x \) is the reduction coefficient which takes into account the effect of hydraulic pressure; \( P \) is the value of hydraulic pressure; \( E \) is elastic modulus; \( R \) and \( t \) are the radius and thickness of the shell respectively, and \( \sigma_s \) is the yield strength.

Equation (3) takes account of plasticity and provides a relation between the shell buckling critical stress and hydraulic pressure. It shows that buckling critical load decreases with increasing pressure. Zhang and Li [32] also carried out an experimental study on plastic buckling of metal cylindrical shells with different sizes; all the shells were filled with water and axially loaded. The results...
showed that the value of buckling critical stress was much smaller than that of classical elastic buckling when hydraulic pressure was rather large, and buckling deformation firstly occurred near the area of boundary and then became axisymmetric, which was similar to the observed instability of large oil storage tanks.

Most literature discussed above focuses on the elastic buckling of thin shells under combined axial compression and internal pressure, and indicates that internal pressure can significantly improve buckling load, which approaches the elastic classical buckling critical load when internal pressure is large enough. Current oil tank design standards adopt these results. Therefore, materials have obvious non-linear effects described by tangent the essence of this phenomenon. In fact, as shown in Fig. 5, steel reduce buckling load in the range of plasticity but does not reveal formula numerically shows that large internal pressure would similar to the observed instability of large oil storage tanks.

Elephant foot buckling critical stress should be relative to these plasticity parameters. If the relation is determined, it is convenient for designers to choose the right steel materials. Moreover, buckling load calculation formulas in present standards do not reflect actual plastic failure mechanism. Rotter's semi-empirical formula numerically shows that large internal pressure would reduce buckling load in the range of plasticity but does not reveal the essence of this phenomenon. In fact, as shown in Fig. 5, steel materials have obvious non-linear effects described by tangent modulus or reduced modulus theory or double-modulus theory. Shanley[34] receive much attention in the past few decades. von Kármán[33] with experimental results well. Shanley buckling loads obtained by the tangent modulus theory agreed well for designers to choose the right steel materials. Moreover, a correction to the present formulas should be made after taking plasticity into consideration. In the following, a detailed theoretical study on the plastic buckling of a large oil tank will be carried out by using the model of Fig. 3. Firstly, we will make a review and some comparisons of theories of plastic buckling in Section 2.2.

2.2. Theories of plastic buckling

Plastic buckling of structures such as bars, plates and shells has received much attention in the past few decades. von Kármán[33] analyzed the stability of a straight bar and obtained the load at which the straight bar became unstable. He assumed, at a given axial load, there existed both loading and unloading portions in the cross section of the bar, therefore the tangent modulus $E_t$ and elastic modulus $E$ were involved. This theory was called the reduced modulus theory or double-modulus theory. Shanley[34] also developed a tangent modulus theory. He considered that no strain reversal occurred at buckling because all bars were geometrically imperfect, whereupon the modulus relating bending strains to bending stresses was only the tangent modulus $E_t$. The buckling loads obtained by the tangent modulus theory agreed well with experimental results well. Shanley’s theory deeply influenced later plastic buckling studies. For the plastic buckling of plates and shells, Hill[35,36] placed the bifurcation criterion for elastic–plastic solids on a firm mathematical foundation, and his theory has formed the basis of practically all investigations of structural buckling in the plastic range[37–42].

According to Hill’s plasticity theory[43], the $J_2$ flow theory gives the increments of strains caused by increments of stresses.

$$\delta\varepsilon_{ij} = \frac{1+\nu}{E} \delta\sigma_i + \frac{1-2\nu}{2E} \Delta\sigma_{ij} \delta\varepsilon_{kk} + \frac{3}{4\nu} \left( \frac{1}{E} - \frac{1}{E_t} \right) S_{ii} S_{il} \delta\sigma_{li} = l_{ijmn} \delta\sigma_{mn}$$

(4)

Here the subscripts range from 1 to 3 represent $i=1,2$; $j=1,2$; $l=1,2$; $m=1,2$. The tensor $\delta\sigma_{ij}$ is the stress deviator and $J_2 = S_{mn} S_{nm}/2$; $\Delta\sigma_{ij}$ is defined as $\Delta\sigma_{ij} = \begin{cases} 1 & i = j; E_i \text{ is the tangent modulus}; \\ 0 & i \neq j \end{cases}$; $\nu$ is Poisson’s ration; $l_{ijmn}$ are called elastic–plastic instability coefficients here.

Based on the model and assumptions in Section 2.1, we can obtain Eq. (5) as follows by expanding Eq. (4).

$$l_{1111} = a_{11} = \frac{1}{E} + \frac{1}{\sigma_i E_t} \left( \sigma_{x} - \frac{1}{2} \sigma_{y} \right)^2$$

$$l_{1122} = a_{12} = \frac{\nu}{E} + \frac{1}{\sigma_i E_t} \left( \sigma_{x} - \frac{1}{2} \sigma_{y} \right) \left( \sigma_{y} - \frac{1}{2} \sigma_{x} \right)^2$$

$$l_{2222} = a_{22} = \frac{1}{E} + \frac{1}{\sigma_i E_t} \left( \sigma_{y} - \frac{1}{2} \sigma_{x} \right)^2$$

(5)

And other components of $l_{ijmn}$ are zero; $\sigma_e = \sqrt{\sigma_{x}^2 - \sigma_{x} \sigma_{y} + \sigma_{y}^2}$ is called equivalent stress.

For the purpose of convenience, we define:

$$\delta\sigma_{ij} = c_{ijmn} \delta\varepsilon_{mn}$$

(6)

then $c_{ijmn} = [l_{ijmn}]^{-1}$; $c_{ijmn}$ are also called elastic–plastic instability coefficients.

2.3. The basic equations

According to Donnell’s simplification[44], the non-linear stability equations used to analyze the buckling of cylindrical shell can be obtained:

Equilibrium equation (symbol “,” means partial derivative):

$$N_{xx} + N_{xy,y} = 0 \quad N_{xy,x} + N_{yy} = 0$$

$$N_{xz,x} + N_{yz,z} + \frac{N_y}{R} + N_x w_{xx} + 2N_{xx}w_{xy} + N_y w_{yy} = P$$

$$M_{xx} + M_{xy,y} - N_{xz} = 0 \quad M_{xy,x} + M_{yy} - N_{yz} = 0$$

(7)

where $N_x, N_y, N_{xz}, N_{yz}$ and $N_{xy}$ are force resultants; $M_x, M_y$ and $M_{xy}$ are moment resultants; $w$ represents the radial displacement.

Internal forces contained in Eq. (7) are:

$$\int_{-t/2}^{t/2} (\sigma_{xy}, \tau_{xy}, \tau_{xz}) dz$$

$$\int_{-t/2}^{t/2} (\tau_{xy}, \sigma_{xy}, \tau_{yz}) dz$$

$$\int_{-t/2}^{t/2} (\sigma_{xz}, \tau_{xz}) dz$$

$$\int_{-t/2}^{t/2} ((\tau_{xy}, \tau_{yz}) dz)$$

(8)
Large deflection non-linear geometric equations:

\[
\begin{align*}
\epsilon_x &= u_x + \frac{1}{2} (w_x)^2 \quad \epsilon_y &= v_y - \frac{w}{R} + \frac{1}{2} (w_y)^2 \\
\gamma_{xy} &= u_y + v_x + w_x w_y \\
\kappa_x &= -w_{xx} \quad \kappa_y = -w_{yy} \quad \kappa_{xy} = -w_{xy}
\end{align*}
\]

(9)

where \(u\) and \(v\) are the shell displacements along the axial and circumferential directions, respectively; \(\epsilon_x\), \(\epsilon_y\) and \(\gamma_{xy}\) are strain components; \(\kappa_x\), \(\kappa_y\) and \(\kappa_{xy}\) are middle surface curvatures of the shell.

Furthermore, as described in Section 2.2, the relationship between stress and strain increments caused by elastic–plastic initial buckling can be expressed as follows:

\[
\frac{\partial \sigma_{ij}}{\partial u_{mn}} = 0 \quad \text{or} \quad \frac{\partial \sigma_{ij}}{\partial u_{mn}} = c_{ijmn} \frac{\partial \epsilon_{mn}}{\partial u_{mn}}
\]

(10)

2.4. Material model

High strength carbon and alloy steels such as Q235B, Q345R and 12MnNiVR are usually used in large oil storage tank wall. These materials have some strength reserve after yielding, as shown in Fig. 5. For example, the tensile strength and yield strength ratio of Q345R is 1.48. Therefore, it is relatively conservative to assume an ideal elastic–plastic material. In this paper, the plastic strengthening effect of the material is taken into consideration and the material is considered as isotropic and power-hardening expressed by the Ramberg–Osgood equation [45].

\[
\epsilon = \frac{\sigma}{E} \left[ 1 + K \left( \frac{\sigma}{\sigma_s} \right)^n \right] \quad n \geq 1
\]

(11)

where \(\sigma\) is uniaxial tensile stress; \(\epsilon\) is uniaxial tensile strain; \(\sigma_s\) is the yield strength of the material; \(K\) and \(n\) are material parameters, based on the material's tensile test.

Differentiating both sides of Eq. (11), we obtain:

\[
\frac{d\sigma}{d\epsilon} = \frac{1}{E} \left[ 1 + K \left( \frac{\sigma}{\sigma_s} \right)^n \right]^{-1}
\]

(12)

Then the tangent modulus is derived.

\[
E_t = \frac{d\sigma}{d\epsilon} = E \left[ 1 + Kn \left( \frac{\sigma}{\sigma_s} \right)^n \right]^{-1}
\]

(13)

2.5. The derivation of elastic–plastic buckling critical calculation formula of perfect tank wall

The model used to analyze plastic buckling failure of a large oil tank is shown in Fig. 3. Assuming pre-buckling deformation of the shell is \((u, v, w)\) and the buckling mode is \((\delta u, \delta v, \delta w)\), the buckling mode should satisfy:

\[
x = 0, \quad L : \delta u_x = 0 \quad \delta v = 0 \quad \delta w = 0 \quad \delta w_{xx} = 0
\]

(14)

\((u, v, w)\) and \((u + \delta u, v + \delta v, w + \delta w)\) are the deformations of two adjacent statuses which suffer the same load, and the corresponding pre-buckling internal forces can be written as \((N_x, N_y, N_z, N_{xz}, M_x, M_y, M_{xy})\). Obviously, \(N_x = \sigma_{0L} t\), \(N_y = \sigma_{0L} t\), and others are zero. The internal forces of the equilibrium configuration adjacent to it are

\[
(N_x + \delta N_x, N_y + \delta N_y, N_{xz} + \delta N_{xz}, N_{yz} + \delta N_{yz}, M_x + \delta M_x, M_y + \delta M_y, M_{xy} + \delta M_{xy}).
\]

Substituting the internal forces and displacements of two statuses into Eq. (7) and (9), and neglecting higher order terms, we derive:

\[
\delta N_{x,x} + \delta N_{x,y} = 0 \quad \delta N_{y,x} + \delta N_{y,y} = 0
\]

\[
\delta N_{x,x} + \delta N_{y,z} + \frac{\delta N_y}{\delta x} + N_x \delta w_{xx} + N_y \delta w_{yy} = 0
\]

\[
\delta M_{x,x} + \delta M_{x,y} - \delta N_{x,y} = 0 \quad \delta M_{y,x} + \delta M_{y,y} - \delta N_{y,x} = 0
\]

(15)

\[
\delta \epsilon_x = \delta u_x \quad \delta \epsilon_y = \delta v_y - \frac{\delta w}{R}
\]

\[
\delta \gamma_{xy} = \delta u_y + \delta v_x
\]

\[
\delta k_x = -\delta w_{xx} \quad \delta k_y = -\delta w_{yy} \quad \delta k_{xy} = -\delta w_{xy}
\]

(16)

Meanwhile, the constitutive equations are:

\[
\frac{\partial \sigma_{ij}}{\partial u_{mn}} = 0 = c_{ijmn} \frac{\partial \epsilon_{mn}}{\partial u_{mn}}
\]

(17)

Combining Eqs. (15)–(17) and Eq. (8), then equilibrium equations which are only expressed by displacements are obtained.

\[
c_{1111} \delta u_{xx} + c_{3333} \delta u_{yy} + (c_{1112} + c_{3333}) \delta u_x - c_{1122} \frac{\delta w_x}{R} = 0
\]

\[
(c_{1112} + c_{3333}) \delta u_x + c_{3333} \delta v_{xx} + c_{2222} \delta v_{yy} - c_{2222} \frac{\delta w_y}{R} = 0
\]

\[
\frac{t^3}{12} \left[ c_{1111} \delta w_{xxxx} + 2 (c_{1112} + 2 c_{3333}) \delta w_{xxyy} + c_{2222} \delta w_{yyyy} \right] + c_{1122} \delta u_x + c_{2222} \left( \delta v_y - \frac{\delta w}{R} \right) - N_x \delta w_{xx} - N_y \delta w_{yy} = 0
\]

(18)

Thus, buckling mode and the corresponding critical load can be determined by seeking the solutions of Eq. (18) satisfying the boundary conditions of Eq. (14).

As described in Section 2.1, both experimental results and actual observations show that the cylindrical shells buckle into an axisymmetric mode under relatively large hydraulic pressure, which is the situation observed in large oil tank when buckling failure occurs. Therefore, in this paper, it is assumed that the buckling mode is axisymmetric, namely, \(\delta u = \delta u(x), \delta v = 0, \delta w = \delta w(x)\). We assume that the solutions of Eq. (18) satisfying the boundary conditions Eqs. (14) take the following form:

\[
\delta u = \sum_{m=0}^{\infty} A_m \sin \frac{m\pi x}{L} = \sum_{m=0}^{\infty} A_m \sin \alpha_m x
\]

\[
\delta v = 0 = \sum_{m=1}^{\infty} C_m \sin \frac{m\pi x}{L} = \sum_{m=1}^{\infty} C_m \sin \alpha_m x
\]

(19)

where \(A_m\) and \(C_m\) are undetermined constants; \(m\) denotes the number of half-waves along the shell length at buckling; \(\alpha_m = m\pi / L\).
Substituting Eq. (19) into Eq. (18), then the second of Eq. (18) is satisfied automatically. From the first of Eq. (14), we obtain:

$$A_m = -\frac{c_{1122}}{\bar{\alpha}_m R C_{1111}} C_m$$  \hfill (20)

From the third of Eq. (14), we derive:

$$\sum_{m=1}^n C_m \left[ \frac{t^2}{12} C_{1111} a_m^4 + \frac{t}{R^2} \left( c_{2222} - c_{1122}^2 \right) C_{1111} + N_s a_m^2 \right] = 0$$  \hfill (21)

Obviously, all \( C_m \) \((m = 1, 2, 3, \ldots)\) cannot be zero at buckling. According to the arbitrariness of \( C_m \), Eq. (22) will be obtained:

$$\frac{t^2}{12} C_{1111} a_m^4 + \frac{t}{R^2} \left( c_{2222} - c_{1122}^2 \right) C_{1111} + N_s a_m^2 = 0$$  \hfill (22)

Substituting \( N_s = \sigma_{s0} \) into Eq. (22), yields:

$$-\sigma_{s0} = \frac{1}{\bar{\alpha}_m R^2} \left[ c_{2222} - c_{1122}^2 \right] C_{1111} + \frac{t^2}{12} C_{1111} a_m^2$$  \hfill (23)

Assuming axial compression to be positive, the buckling critical stress of the shell can be determined with \( d \sigma_{s0}/d \alpha_m = 0 \).

$$\sigma_{cr} = \sqrt{\frac{3t}{3R}} \sqrt{C_{1111} c_{2222} - c_{1122}^2}$$  \hfill (24)

Considering \([c_{ijmm}]/[l_{ijmm}] = 1\), Eq. (24) becomes:

$$\sigma_{cr} = \sqrt{\frac{3t}{3R}} \sqrt{\alpha_{1111} l_{2222} - l_{1122}^2} = \sqrt{\frac{3t}{3R}} \frac{1}{\alpha_{11} \alpha_{22} - \alpha_{12}^2}$$  \hfill (25)

where:

$$a_{11} = 1 + \frac{1}{\bar{\alpha}_e} E - \frac{E}{E_l} \left( \sigma_x + \frac{1}{2} \sigma_y \right)^2$$

$$a_{12} = -\frac{t}{\bar{\alpha}_e} \frac{1}{\frac{E}{E_l} - 1} \left( \sigma_x + \frac{1}{2} \sigma_y \right) \left( \sigma_y + \frac{1}{2} \sigma_x \right)$$

$$a_{22} = 1 + \frac{1}{\bar{\alpha}_e} E - \frac{E}{E_l} \left( \sigma_y + \frac{1}{2} \sigma_x \right)^2$$

(26)

2.6. The buckling critical stress calculation formula of tank wall considering the effect of material plasticity

When the material is in the elastic state, \( E_l = E \). Setting \( v = 0.3 \) and substituting this into Eqs. (25),(26), the result is:

$$\sigma_{cr} = \sqrt{\frac{3}{3v^2-1}} \frac{E_l}{R} = 0.605 \frac{E_l}{R} = 1.21 \frac{E_l}{D}$$  \hfill (27)

Eq. (27) is the elastic classical buckling critical stress [31] of the perfect cylindrical shell.

Buckling critical stress calculation formulas of an oil tank wall in current seismic design standards such as API 650, JIS B 8501 and GB 50341 can be unified as Eq. (28).

$$\sigma_{cr} = \frac{C E_l}{D}$$  \hfill (28)

The coefficient \( C \) in API 650, JIS B 8501 and GB 50341 is 0.413, 0.33 and 0.15, respectively. Eq. (28) is obtained by setting a certain safety coefficient on Eq. (27) to consider the effect of initial geometric imperfections, variable thickness and geometric non-linearity. For example, the safety coefficient is 3.0 in API 650. According to API 650, the allowable stress of steel at normal temperature is set to be \( 2 \sigma_y / 3 \) when calculating the thickness of the tank wall, thus the ratio of circumferential stress and material yield strength is \( PR/\sigma_y = \sigma_r/\sigma_y = 2/3 \) in the static state. Namely, in fact, Eq. (28) is available for relatively large value of \( PR/\sigma_y \).

It is obvious that buckling critical stress formulas in the present tank seismic design standards are derived on the basis of elastic buckling theory of cylindrical shell. However, the material of tank wall has already entered the plastic phase when elephant foot buckling occurs. Therefore, the effect caused by the plastic properties of material on the buckling strength can’t be neglected, and an effective way is to make a plasticity correction on Eq. (28).

Eq. (25) is available for a perfect cylindrical shell and can be used to calculate plastic buckling critical stress. Similarly, by introducing a certain safety coefficient, we obtain:

$$\sigma_{cr} = CM_p \frac{E_l}{D}$$  \hfill (29)

where \( M_p = 0.954 \sqrt{1/E_l^2 (\sigma_{11}^2 - \sigma_{12}^2)} \) is called the plasticity correction coefficient and the value of \( C \) is set according to the relevant standards. When the material is in the state of elasticity, \( E_l = E \) and \( M_p = 1.0 \).

When elephant foot buckling occurs, the circumferential stress of the tank wall is close to or larger than the yield stress of material, then \( \sigma_y >> \sigma_x \). Therefore, Eq. (29) can be simplified further. Neglecting \( \sigma_x \) in Eq. (26) and setting \( v = 0.3 \), we have:

$$M_p = 1.908 \frac{E_l}{3.8E_l - 0.16E_l}$$

(30)

Formula (29) is the plastic buckling critical stress calculation formula of tank wall. In general, when the tank wall material is in the state of elasticity, the critical load can be calculated by Eq. (28). Once the material yields, according to Eq. (13), tangent modulus \( E_l \) will decrease rapidly. In this situation, the effect caused by material

<table>
<thead>
<tr>
<th>Material</th>
<th>Yield strength/MPa</th>
<th>Tensile strength/MPa</th>
<th>Elastic modulus/MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q235A</td>
<td>245</td>
<td>382</td>
<td>204,875</td>
</tr>
<tr>
<td>Q345R</td>
<td>370</td>
<td>573</td>
<td>207,900</td>
</tr>
</tbody>
</table>

Fig. 6. The curves of material properties.
plasticity on the buckling critical stress of tank wall must be taken into consideration.

As the tangent modulus $E_t$ is usually much less than the elastic modulus $E$, simplifying Eq. (29) with the consideration that $3.8E \gg 0.16E_t$ yields:

$$\sigma_{cr} = CM_p \frac{E_t}{D}$$  \hspace{1cm} (31)

where $M_p = 0.979 \sqrt{\frac{E_t}{E}}$.

3. Illustrative calculation examples and comparisons

Q235B and Q345R are the main materials for manufacturing large oil tanks at present in China. In this section, calculations and comparative studies on the buckling critical stress of a tank wall made of Q235B and Q345R under combined high hydraulic pressure and axial compression will be carried out. Tensile tests of Q235B and Q345R have been done in order to obtain the accurate curves of material properties, and the test data are listed in Table 1.

By using the least square method for fitting test data, we obtain:

$$\varepsilon = \frac{\sigma}{204875} \left[1 + \frac{0.412957}{245} \frac{\sigma}{\sigma_y}\right]^{12}$$

$$\varepsilon = \frac{\sigma}{207900} \left[1 + \frac{1.061667}{370} \frac{\sigma}{\sigma_y}\right]^{9}$$  \hspace{1cm} (32)

For Q235B and Q345, respectively, both actual test and fitting curves of material stress–strain relations are plotted in Fig. 6. We can find from Fig. 6 that a certain error exists near the area of yield point for the two curves. After further comparison, it is found that, in most cases the value of the tangent slope of the fitting curve is smaller than that of the test curve, then the results of buckling critical stress obtained from Eq. (29) is more conservative. Combining Eq. (29) and (32), the buckling critical stress calculation formula of a tank wall under the high hydraulic pressure can be obtained.

There are two tanks with different materials and sizes. The material of one is Q235B, and the volume is $5 \times 10^3$ m$^3$ (the geometric parameters listed in Table 2 are from GB50341). The material of the other one is Q345R, and the volume is $5 \times 10^4$ m$^3$.

According to GB 50341, API 650, JIS B 8501, Eq. (29),(31) (C = 0.413, the same below) and EN 1993-1-6 (2006), the buckling critical stress calculation results are plotted in Figs. 7 and 8, respectively. From Figs. 7 and 8 (The horizontal axis is the dimensionless circumferential stress $PR/t_0$ and the vertical axis is the ratio of calculation buckling critical stress and classical elastic buckling critical stress $\sigma_{cr}/\sigma_{cl}$), we can find that:

<table>
<thead>
<tr>
<th>Material</th>
<th>Volume/m$^3$</th>
<th>Radius/m</th>
<th>Thickness/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q235A</td>
<td>$5 \times 10^3$</td>
<td>11</td>
<td>0.0115</td>
</tr>
<tr>
<td>Q345R</td>
<td>$5 \times 10^4$</td>
<td>30</td>
<td>0.0315</td>
</tr>
</tbody>
</table>

![Fig. 7. Comparisons of buckling critical stresses of tank wall made of Q235B ($R/t = 956.52$).](image)

![Fig. 8. Comparisons of buckling critical stresses of tank wall made of Q345R ($R/t = 952.38$).](image)
(1) Eq. (31) is the simplified result of Eq. (29) under high hydraulic pressure. From Figs. 7 and 8, it is indicated that if the material of the tank wall is Q345R, the difference between Eqs. (31) and (29) is small under the condition of \( \frac{pR}{t_{os}} \geq 0.6667. \) However, if the material is Q235B, the difference is slightly greater when the hydraulic pressure is not very large. Eq. (31) is very convenient to be used for calculating plastic buckling critical stress of tank wall. The change of curves is similar to the European steel shell design standard formula (Rotter’s semi-empirical formula).

(2) According to the results calculated by Eqs. (29), (31) and Rotter’s semi-empirical formula, as shown in Figs. 7 and 8, it can be concluded that with the increase of hydraulic pressure, the tank wall material changes from elasticity to plasticity, and buckling load of tank wall decreases rapidly. Because current seismic design standards of oil tank can’t reflect the effect caused by material plasticity on buckling critical stress, plasticity effect correction is necessary.

(3) Both Eqs. (31) and (3) indicate that under relatively large hydraulic pressure, the buckling critical stress decreases rapidly with the increase of hydraulic pressure. However, Eq. (3) is just a fitting formula which reflects such changes in value. Eq. (31) can reveal this phenomenon essentially, that is, the tangent modulus of the tank wall material reduces rapidly after the material enters the plastic zone.

4. Conclusions and prospects

(1) A study on the elastic—plastic stability of a large oil storage tank wall under the interaction of high hydraulic pressure and axial compression has been carried out. By introducing plasticity effect coefficient, some corrections are performed on the buckling critical stress calculation formulas of the tank wall in main seismic design standards in the world such as API 650, JIS B 8501 and GB 50341. A formula which is available for engineering application and considers the actual failure mechanism of the tank is obtained, and it can provide aid for further protection of a tank.

(2) Under the effect of high hydraulic pressure caused by earthquake, the buckling strength of a tank wall will decrease rapidly due to rapid material property changes when it enters the plastic zone. It can be seen that the strengthening effect of the material with larger ratio of tensile strength and yield strength will be more obvious, and the loss of the buckling strength of tank wall made of this material will be smaller. This can provide some indications for material options.

(3) From the perspectives of engineering application, simplified analyses of the plastic stability of a tank which suffers the interaction of high hydraulic pressure and axial pressure have been discussed in this paper. In fact, plastic buckling of cylindrical shells is very complicated and has not been solved perfectly. Disputes still exist between incremental theory and full deformation theory. Geometric non-linearity can cause complex changes of post-buckling behaviors. The analysis of coupling effect of geometric non-linear, material non-linear and initial geometric imperfections still faces theoretical difficulties, therefore further study is necessary for the buckling problem of tank.

Funding

This work was supported by the National Key Technologies Research and Development Program of China (grant number 2011BAK06B02), the National High Technology Research and Development Program of China (863 Program, No. 2012AA040103) and the Key Science and Technology Innovation Team Project of Zhejiang Province, China (grant number 2010R50001).

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